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# Nonlinear temporal-spatial modulation of near-planar Rayleigh waves in shear flows: formation of streamwise vortices

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The nonlinear temporal-spatial modulation of a near-planar Rayleigh instability wave is studied. The amplitude of the wave is allowed to be a slowly varying function of spanwise position as well as of time (or streamwise variable in the spatial evolution case). It is shown that the development of the disturbance is controlled by critical-layer nonlinear effects when the linear growth rate decreases to $O(\epsilon^{\frac{2}{3}})$, where $\epsilon$ is the magnitude of the disturbance. Nonlinear interactions influence the evolution by producing spanwise dependent mean-flow distortions. The evolution is governed by an integro-*partial*-differential equation containing history-dependent nonlinear terms of Hickernell (1984) type. A notable feature of the amplitude equation is that the highest derivative with respect to spanwise position appears in the nonlinear terms. These terms are associated with three-dimensionality. The possible properties of the amplitude equation are discussed. Numerical solutions show that a disturbance initially centred at a spanwise position can propagate laterally to form concentrated, quasi-periodic streamwise vortices. This qualitatively captures the phenomena observed in experiments. The focusing of vorticity may be associated with a localized singularity which can occur at a finite distance downstream or within a finite time. It is noted that the amplitude equation is rather generic and applies to a broad class of shear flows which is inviscidly unstable.

## 1. Introduction

### 1.1. *Streamwise vortices in boundary layers*

Experiments on boundary-layer transition have revealed that following an initial two-dimensional development stage, three-dimensional disturbances grow quickly. Consequently, the mean-flow is distorted into a 'peak-valley' splitting pattern consisting of streamwise (longitudinal) vortices. High shear layers appear in the valleys, where secondary small-scale instability occurs accompanied by the eruption of vorticity from the wall layer, e.g. see Klebanoff, Tidstrom & Sargent (1962). Direct numerical simulations have confirmed these observations (see e.g. Kleiser & Zang 1991). A comprehensive list of experimental and computational studies can be found in e.g. Hall & Smith (1990, 1991). Similar events have also been observed in fully developed turbulent boundary layers (e.g. Kline *et al.* 1968; Kim, Kline & Reynolds 1971).

Such phenomena were studied theoretically by Stuart (1984, 1987) from the viewpoint of initial value problems. The streamwise vortices were modelled by an inviscid secondary flow embedded in an otherwise parallel shear flow. Assuming that the secondary flow is symmetric in the spanwise direction, Stuart (1990) showed that a singularity may occur on the symmetric plane in a finite time. The singularity is of unsteady separation type (see e.g. Cowley, Van Dommelen & Lam 1990). In Stuart's theory, the secondary flow does not have to be associated with any instability waves. While being a highly simplified model, it incorporates two fundamental aspects: three-dimensionality and nonlinearity, which are believed to be prerequisites for the flow to undergo stretching and tilting. Studies which do not invoke instability modes include, Landahl (1975), Breuer & Landahl (1990) and Breuer & Haritonidis (1990). They suggest that the transient response of a parallel shear flow to three-dimensional perturbations may be more important than instability modes in certain circumstances, such as in so-called 'by-pass' transition.

An alternative approach, which is based on hydrodynamic instability, is to study the spatially modulated Tollmien–Schlichting (T-S) waves (Hocking, Stewartson & Stuart 1972; Davey, Hocking & Stewartson 1974). This idea has recently been developed by Smith & Walton (1989) and Hall & Smith (1990) in the framework of high-Reynolds-number asymptotic approximations. They considered the nonlinear evolution of near-planar waves which are slightly warped in the spanwise direction. The lower-branch scaling regime was adopted. It was shown that a small three-dimensional 'warping' could be amplified by nonlinear effects leading to spanwise concentration of streamwise vorticity and that a large mean-flow distortion could be provoked by relatively small-amplitude disturbances (see §6.2). The dominant interaction was between the wave and the induced mean-flow distortion while the harmonics did not play a significant role. More recently, Stewart & Smith (1992) and Smith & Bowles (1992) considered similar problems in the so-called 'high-frequency' limit of the lower-branch regime. It is claimed that the dominant nonlinear effects first become important in the bulk of the flow while the critical layers remain passive. Comparisons with experiments were made and were found to be favourable.

### 1.2. *Streamwise vortices in free shear layers and Stokes layers*

It has been well recognized that streamwise vortices also exist in free shear layers; see e.g. Konrad (1976), Breidenthal (1981), Bernal (1981), Jimenez (1983), Jimenez, Cogollos & Bernal (1985). They observed that longitudinal streaks or pairs of counter-rotating streamwise vortices were superimposed on the primary spanwise vortices. Bernal (1981) and Bernal & Roshko (1986) suggest that the counter-rotating vortex pair are part of a vortex that continuously loops back and forth in the region between adjacent spanwise vortices. Experimental studies of the origin and the evolution of the streamwise vortices include Lasheras, Cho & Maxworthy (1986), Lasheras & Choi (1988), Bell & Mehta (1992). They suggest that small streamwise vorticity is stretched by the strain produced by the primary spanwise vortices to form concentrated vortices – a mechanism proposed earlier by Lin & Corcos (1984). The vortices were observed to appear first in the braid region, i.e. the region where the spanwise vorticity is minimum. This led them to suggest that the stretching mainly takes place there. Nygaard & Glezer (1991), on the other hand, suggest that longitudinal vortices arise as a result of instability of the primary spanwise vortex core. The formation of streamwise vortices was also attributed to secondary instability. We shall discus these aspects in §7.

Lasheras, Choi & Maxworthy (1986, hereafter referred as LCM) studied how streamwise vortices developed from a localized perturbation in an otherwise two-dimensional flow. They found that as the localized disturbance was convected downstream, it spread laterally and ultimately evolved into a rather regular configuration consisting of counter-rotating vortices. The lateral propagation exhibited a 'wave-like' feature. The streamwise location at which the longitudinal vortices first appeared depended on the magnitude of the upstream disturbance and the position where it was introduced. A mechanism for the 'lateral spreading' was proposed by them. In this paper, we shall develop a theory which will be able to capture this behaviour qualitatively. In related experiments, Lasheras & Choi (1988) and Nygaard & Glezer (1991) showed that a weak, spanwise-periodic disturbance introduced upstream could also develop into concentrated streamwise vorticity; our approach applies to this situation as well provided that the spanwise lengthscale is large.

There have been several numerical and theoretical studies of these phenomena observed in free shear layers. Lin & Corcos (1984) argued that the local flow in the braid region could be approximated by a uniformly straining field. They thus studied the evolution of spanwise-periodic vorticity in such a flow, and found that concentrated streamwise vortices could form under the combined effects of straining and nonlinearity. Pullin & Jacobs (1986) further studied the Lin–Corcos model using a contour-dynamics method. However, in the Lin–Corcos model, while the uniform strain field may well represent the local effect of the spanwise vortices, the shear, i.e. the spanwise vorticity of the basic flow, is completely ignored. A more realistic model is that of Ashurst & Meiburg (1988), in which the shear layer was idealized as a vortex sheet. The three-dimensional disturbance was introduced by dislocating the centreline of each spanwise vorticity filament periodically. The temporal sequences computed by them were able to capture certain features observed by Lasheras & Choi (1988). Ashurst & Meiburg also considered the evolution of a localized disturbance, and found that it could propagate laterally as observed by LCM. However they did not preclude the possibility that the result was due to the spurious instability of the filament model. On the other hand, Pierrehumbert & Widnall (1982) identified the so-called translative instability mode for Stuart vortices. They suggested that the observed three-dimensional structure might be related to this mode.

Longitudinal vortices were also observed in oscillatory Stokes layers. Hino, Sawamoto & Takasu (1976) inferred their existence from the measurement of velocity profiles. Subsequently, Hino *et al.* (1983) visualized the structure of the vortices and found that they were similar to those observed in the turbulent boundary layers. On the theoretical side, the instability analysis for time-dependent shear flows (e.g. Stokes layers) of high-Reynolds-number is parallel to that for spatially evolving shear layers; see e.g. Cowley (1987) and Wu (1991).

### 1.3. *The scope of the present study*

This paper is to investigate vorticity concentration in shear flows, particularly in free shear layers. The instability waves of concern are *Rayleigh modes.* Following the idea of Hocking *et al.* (1972) and recent work of Smith & Walton (1989) and Hall & Smith (1990) on T-S waves, we consider a disturbance which is basically two-dimensional but is slowly modulated in the spanwise direction. This modulation represents some unavoidable three-dimensional distortion or 'imperfection' present in the flow. We shall examine whether such a slight imperfection can be amplified by nonlinear effects. A number of experiments suggest that the streamwise vortices appear to be 'facility induced', i.e. they result from amplification and redistribution of some small imperfections (Jimenez 1983; Jimenez *et al.* 1985; LCM; Lasheras & Choi 1988; Nygaard & Glezer 1991).

As will be shown, the crucial nonlinear effect is associated with the unsteady (and viscous) *critical layers*, i.e. thin regions centred on levels at which the phase velocity of the disturbance is equal to the basic flow velocity. This is in contrast to the theoretical studies mentioned above, where such an effect is absent (or not investigated). Nonlinear effects of unsteady critical layers on instabilities of flows have been studied, e.g. by Hickernell (1984), Churilov & Shukhman (1988), Goldstein & Leib (1989), Goldstein & Choi (1989), Shukhman (1991), Goldstein & Lee (1992), Wu (1992), Wu, Lee & Cowley (1993). A common feature is that nonlinear interactions within the critical layers can control the overall development of the disturbances. The resulting amplitude equations contain history-dependent nonlinear terms. In particular, for a pair of oblique modes, Goldstein & Choi (1989) showed that because of a simple pole (singularity) in the streamwise and spanwise velocities of the disturbance, the evolution is much more sensitive to nonlinear effects than the purely two-dimensional modes. This indicates the importance of three-dimensionality in the nonlinear stage. Wu *et al.* (1993) extended Goldstein & Choi's analysis into the viscous regime. By analysing the very viscous limit, they demonstrated that a link can be established between the unsteady-critical-layer approach of Goldstein and co-workers and the wave–vortex interaction approach of Hall & Smith (1991) and Brown, Brown & Smith (1993). In particular, in the viscous limit the amplitude equation of Wu *et al.* (1993) reverts to that of Smith, Brown & Brown (1993). However, in all previous studies of unsteady critical layers, the disturbances are assumed to be either two-dimensional or strictly periodical in the spanwise direction. Here we shall relax this restriction by allowing the spanwise distribution to modulate so that we can investigate vorticity concentration in space.

The basic flow that we choose to present our analysis is a Stokes layer generated by a flat plate oscillating sinusoidally in an infinite fluid. The reason is that for the Stokes layer the critical layers are not necessarily located at inflexion points. Thus we can present our theory in a more general form. We shall show that free shear layers can be treated as a special case, though some minor modification to the analysis is necessary because the evolution is now spatial rather than temporal. The linear instability of the Stokes layer has been studied by Tromans (1979) and Cowley (1987). Nonlinear effects have been studied in three previous papers (Wu & Cowley 1993; Wu *et al.* 1993; Wu 1992) for two-dimensional waves, pairs of oblique waves and resonant triad waves.

As in Wu & Cowley (1993), to fix our ideas we consider the modes existing beneath the solid curve A in figure 1. In particular, suppose that an instability wave with wavenumber $\alpha$ is excited at time $\tau_i$ on the left branch of the neutral curve A. We assume that the dependence on the spanwise location is weak so that the dispersion relation is not affected at leading order by this weak three-dimensionality. According to linear theory (Tromans 1979; Cowley 1987), the disturbance grows exponentially until the neutral time, say $\tau_0$ on the right branch† of neutral curve A, is approached. Because the linear growth rate is small, critical layers emerge. Nonlinearity, among other effects, may become important (Maslowe 1986; Stewartson 1981). The main concern of this study is how *critical-layer nonlinearity* affects the development of such a small disturbance, in particular how the disturbance vorticity is redistributed in space.

The overall plan of the paper is as follows. In §2, we analyse nonlinear interactions inside the critical layers to fix the underlying scalings. In §3, we consider the flow

† We note that our analysis also applies to the vicinity of the left branch of the neutral curves A and B. A discussion of when nonlinearity becomes important in these regions follows straightforwardly from that of Wu *et al.* (1993).

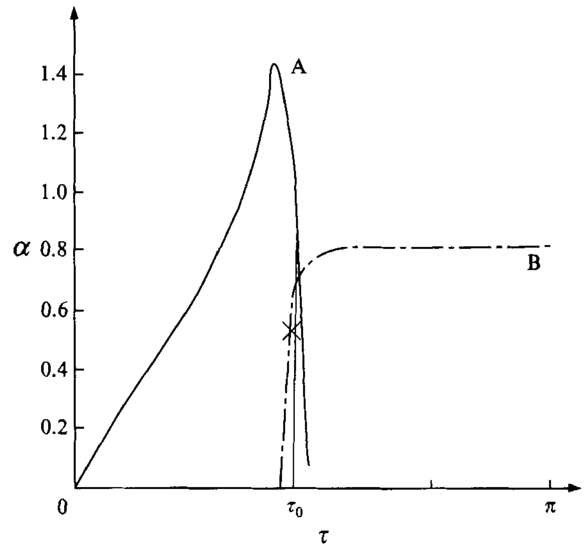


FIGURE 1. Sketch of the linear neutral diagram for wavenumbers $\alpha$ (from Cowley 1987). $\tau_0$ is a point on the right-hand branch of the neutral curve A. In the analysis we concentrate on times close to $\tau = \tau_0 + \epsilon^{\frac{2}{3}}\tau_1$, where $\epsilon$ is the magnitude of the disturbance and $\tau_1$ is an order-one number.

in the 'outer' region away from the critical layers. The asymptotic solutions near the critical layers are constructed, which as usual contain some undetermined jumps. A solvability condition is deduced for an inhomogeneous Rayleigh equation. In §4, the inner expansions are carried out and the solutions which match with the outer expansions are found. By matching, we determine the jumps. In §5, using the solvability condition and the jumps, we obtain the amplitude equation for the case of the Stokes layers. The amplitude equation for free shear layers is given in Appendix A, where we modify the analysis presented in the main text of the paper. In §6, we discuss the properties of the amplitude equation. Numerical solutions are presented and related to experiments. Finally in §7 we summarize the results of this study, and make some comments.

## 2. Scaling arguments and formulation

We take the flow to be described by Cartesian coordinates $(x^*, y^*, z^*)$, where $x^*$ is parallel to the direction of oscillation of the plate, $y^*$ is normal to the plate and $z^*$ is the spanwise direction. The velocity of the flow on the boundary $y^* = 0$ is $(U_0 \cos \omega t^*, 0, 0)$, where $t^*$ is the dimensional time. The Stokes layer has a thickness $\delta = (2\nu/\omega)^{\frac{1}{2}}$, where $\nu$ is the kinematic viscosity. The Reynolds number based on $\delta$ is $R = (2U_0^2/\omega\nu)^{\frac{1}{2}}$. We non-dimensionalize the time with $\omega^{-1}$, i.e. $\tau = \omega t^*$, and write $(x^*, y^*, z^*) = \delta(x, y,, z)$ and the velocity as $U_0(U, V, W)$. Then the basic flow field is

$$(\bar{U}, \bar{V}, \bar{W}) = (\cos(\tau - y)\mathrm{e}^{-y}, 0, 0) \ .$$

We denote the perturbed flow by

$$(\bar{U} + u, v, w) \ .$$

Following Tromans (1979), we study high-frequency instability waves, and introduce the fast timescale

$$t = \tfrac{1}{2}R\tau\ . \tag{2.1}$$

The normal velocity component $v$ of a small-amplitude disturbance, to the leading order, has the form

$$\epsilon A_0 \bar{v}(y,\tau)E\cos\beta z + \text{c.c.}\ ,$$

where $\epsilon \ll 1$, and $\beta$ is the spanwise wavenumber; we assume that $\beta \ll 1$. Hereafter, c.c. represents the complex conjugate. For the purpose of deriving the scaling, it is sufficient to write the dependence on $z$ in this special form. But from the next section, a general dependence on the slow spatial variable $Z = \beta z$ will be allowed. The function $\bar{v}$ satisfies the Rayleigh equation, and the order-one complex constant $A_0$ is a measure of the scaled amplitude of the disturbance. For convenience, we have defined

$$E = \exp(\mathrm{i}\alpha x - \mathrm{i}\hat{\theta}(t))\ , \quad \frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t} = \alpha c(\tau) + R^{-\frac{1}{2}}\Omega_1(\tau) + \dots\ , \tag{2.2}$$

where $c$ is a complex phase velocity, i.e. $c = c_r + \mathrm{i}c_i$, and $\alpha$ is the wavenumber. In this study, we assume that $\alpha$ is of order one. The $O(R^{-\frac{1}{2}})$ correction to the local frequency in (2.2) is from the viscous sub-Stokes layer adjacent to the wall (Cowley 1987).

As discussed earlier, critical-layer nonlinearity asserts its influence at the times near to the neutral time, where the linear growth rate $\alpha c_i \ll 1$. Before we can perform a formal asymptotic expansion, we need to specify the evolution timescale, the thickness of the critical layers, and the spanwise scale $\beta$ in terms of $\epsilon$. To this end, we introduce an intermediate timescale $t_1 = \alpha c_i t$ to take account of the slow growth of the amplitude. This timescale is much 'slower' than that of the instability waves, but much 'faster' than the timescale of the underlying Stokes flow. The disturbance is then described by the time variables $t$, $t_1$, $\tau_1$ and $\tau$. In terms of these new variables, the $x$-momentum equation can be written as follows:

$$\left\{\alpha c_i\frac{\partial}{\partial t_1}+(\bar{U}-c_r)\frac{\partial}{\partial x}\right\}u+v\frac{\partial \bar{U}}{\partial y}-\frac{1}{R}\Delta u = -\frac{\partial p}{\partial x}-\frac{\partial u^2}{\partial x}-\frac{\partial uv}{\partial y}-\frac{\partial uw}{\partial z}\ , \tag{2.3}$$

where $\partial/\partial t$ has been replaced by $-c_r\partial/\partial x$ after using (2.2) since this is accurate to the order required, and $\Delta$ is the Laplacian operator.

Suppose that a critical layer is located at $y_c$, i.e. $\bar{U}(y_c) = c_r$ and has a thickness of $O(\mu)$, then near the critical layer $(\bar{U} - c_r)$ is of order $\mu$ if $\bar{U}_y(y_c) = O(1)$. We are particularly interested in the case where the time-variation term appears at leading order in the critical-layer equations. This requires $\alpha c_i \partial u/\partial t_1$ to balance $(\bar{U} - c_r)\partial u/\partial x$, which gives

$$\mu \sim O(c_i)\ . \tag{2.4}$$

Such a scaling brings in a non-equilibrium effect that does not occur in the analysis of, for example, Davey *et al.* (1974), Smith & Walton (1989). It leads to a different type of amplitude equation that contains history-dependent nonlinear terms of Hickernell (1984) type.

To ensure that a spatial modulation term $\partial^2/\partial Z^2$ appears in the final amplitude equation, we require that

$$\beta^2 \sim O(\alpha c_i)\ . \tag{2.5}$$

The asymptotic property of the Rayleigh equation tells us that $v \sim O(\epsilon)$ as $y \to y_c$. The asymptotic behaviour of the cross-flow component $w$ is $w \sim \epsilon\beta(y - y_c)^{-1}$. It follows from the continuity equation that

$$u \sim \epsilon\beta^2(y-y_c)^{-1} + \epsilon\bar{U}_{yy}(y_c)\log(y-y_c) \ . \tag{2.6}$$

Owing to three-dimensionality, the singularity appears as a simple pole. The logarithmic singularity is associated with $\bar{U}_{yy}(y_c) \neq 0$. It turns out that as far as deriving the scaling is concerned, we can concentrate only on the pole type of singularity without losing generality. This is because the resultant scaling happens to take the logarithmic singularity into account as well (see below).

To fix the scaling, we analyse nonlinear interactions within the critical layers. The quadratic interaction, typically through $vu_y$, produces a forcing of $O(\epsilon^2\beta^2\mu^{-2})$. Balancing this forcing term with $\alpha c_i \partial u/\partial t_1$ in (2.3), we find that it generates a harmonic and a mean-flow distortion, say $u^{(2)}$, of order $(\epsilon^2\beta^2\mu^{-3})$. The cubic interaction through $vu_y^{(2)}$ produces a forcing of $O(\epsilon^3\beta^2\mu^{-4})$. Balancing this forcing term with $\alpha c_i \partial u/\partial t_1$, we conclude that it drives a fundamental $u^{(3)}$ of $O(\epsilon^3\beta^2\mu^{-5})$. In order for the evolution of the disturbance to be affected by these interactions, this nonlinearly generated fundamental is required to match with the unsteadiness of $O(\alpha c_i \epsilon) \sim O(\mu\epsilon)$ in the outer region (cf. Hickernell 1984), i.e.

$$O(\epsilon^3\beta^2\mu^{-5}) \sim O(\epsilon\mu) \ . \tag{2.7}$$

Since $\alpha \sim O(1)$, from (2.4), (2.5) and (2.7), we obtain

$$\beta \sim O(\epsilon^{\frac{1}{5}}) \ , \tag{2.8}$$

$$\mu \sim O(\epsilon^{\frac{2}{5}}) \ . \tag{2.9}$$

Finally, we check for the effect of viscosity. A distinguished case is that the viscous diffusion terms appear at leading order in the critical-layer equations. This occurs for $R^{-\frac{1}{3}} \sim O(\mu)$, i.e. $R \sim O(\epsilon^{-\frac{6}{5}})$. So we write

$$R^{-1} = \lambda\epsilon^{\frac{6}{5}} \ , \tag{2.10}$$

where the parameter $\lambda$ is introduced to reflect viscosity effects (cf. Haberman 1972). It will be assumed to be order one in §§3–5. The very viscous case corresponding to $\lambda$ being asymptotically large will be discussed in §6.

Equations (2.8), (2.9) and (2.10) set the underlying scalings for this study. Note that (2.9) shows that the nonlinear evolution timescale happens to be the same as that for a purely two-dimensional wave with a logarithmic singularity (cf. Hickernell 1984; Wu & Cowley 1993). However, this is merely a coincidence because the present scaling is obtained solely by considering the pole type of singularity, irrespective of the presence of the logarithmic branch point. Nevertheless, under the present scaling, the nonlinear effect associated with the latter singularity is also incorporated and it is expected that a corresponding nonlinear term with a coefficient proportional to $\bar{U}_{yy}(y_c)$ will appear in the final amplitude equation (see §5).

Equations (2.4) and (2.9) indicate that we should concentrate on times near to

$$\tau = \tau_0 + \epsilon^{\frac{2}{5}}\tau_1 \ , \tag{2.11}$$

i.e. the times at which the linear growth rate is $O(\epsilon^{\frac{2}{5}})$. It is at this stage that nonlinear effects control the development of the disturbance, and redistribute the disturbance vorticity. For convenience, the nonlinear evolution timescale is rewritten as

$$t_1 = \epsilon^{\frac{2}{5}} t \ . \tag{2.12}$$

The evolution of the disturbance is thus described by four time variables, namely $t$,

$t_1$, $\tau_1$ and $\tau_0$; the time derivative $\partial/\partial\tau$ is transformed according to

$$\frac{\partial}{\partial \tau} \rightarrow \frac{1}{2} R \frac{\partial}{\partial t} + R\epsilon^{\frac{2}{5}} \frac{\partial}{\partial t_1} + \epsilon^{-\frac{2}{5}} \frac{\partial}{\partial \tau_1} + \frac{\partial}{\partial \tau_0} \, . \tag{2.13}$$

Since the basic flow $\bar{U}$ evolves on the slow time variable $\tau$, it is sufficient to expand its profile at time $\tau$ in a Taylor series about the neutral time $\tau_0$:

$$\bar{U}(y,\tau) = \bar{U}(y,\tau_0) + \epsilon^{\frac{2}{5}} \bar{U}_\tau(y,\tau_0)\tau_1 + \dots \, .$$

Hereafter all the quantities associated with the basic flow will be evaluated at $\tau_0$ unless otherwise mentioned.

To take account of spanwise modulation, we introduce a slow spatial variable

$$Z = \epsilon^{\frac{1}{5}} z \, , \tag{2.14}$$

and thus

$$\frac{\partial}{\partial z} \rightarrow \epsilon^{\frac{1}{5}} \frac{\partial}{\partial Z} \, . \tag{2.15}$$

## 3. Outer expansion

### 3.1. *Asymptotic solutions near the critical layers*

To the order of approximation required in this study, the flow outside the critical layers is linear and inviscid. The velocity and the pressure of the disturbance are expanded as

$$u = \epsilon u_1 + \epsilon^{\frac{6}{5}} u_2 + \epsilon^{\frac{7}{5}} u_3 + \dots \, , \tag{3.1}$$

$$v = \epsilon v_1 + \epsilon^{\frac{6}{5}} v_2 + \epsilon^{\frac{7}{5}} v_3 + \dots \, , \tag{3.2}$$

$$w = \epsilon^{\frac{6}{5}} w_1 + \epsilon^{\frac{7}{5}} w_2 + \epsilon^{\frac{8}{5}} w_3 + \dots \, , \tag{3.3}$$

$$p = \epsilon p_1 + \epsilon^{\frac{6}{5}} p_2 + \epsilon^{\frac{7}{5}} p_3 + \dots \, . \tag{3.4}$$

The 'earlier time' linear solutions suggest that $v_1$ can be written as

$$v_1 = A(Z, t_1, \tau_1, \tau_0)\bar{v}_1(y, \tau_0)E + \text{c.c.} \, , \tag{3.5}$$

where $A(Z, t_1, \tau_1, \tau_0)$ is the amplitude, and now is allowed to be an arbitrary function of $Z$. Here $E$ is defined by (2.2) except that $\tau$ is replaced by $\tau_0$ since we concentrate on the vicinity of the neutral time. As in Wu & Cowley (1993), the dependence on $\tau_0$ and $\tau_1$ is parametric to the order of interest in this study, and will not be written out explicitly hereafter. The function $\bar{v}_1$ satisfies the Rayleigh equation

$$(\bar{U} - c)(\mathrm{D}^2 - \alpha^2)\bar{v}_1 - \bar{U}_{yy}\bar{v}_1 = 0 \, , \quad \text{with} \quad \mathrm{D}^2 = \frac{\mathrm{d}^2}{\mathrm{d}y^2} \, . \tag{3.6}$$

The boundary conditions are that $\bar{v}_1 = 0$ on $y = 0$, and $\bar{v}_1 \rightarrow 0$ as $y \rightarrow +\infty$.

Let $\eta = y - y_c^j$, where $y_c^j$ is the $j$th critical level at which $\bar{U} = c$. Then as $\eta \rightarrow \pm 0$, $\bar{v}_1$ has the asymptotic behaviour

$$\bar{v}_1 \rightarrow a_j^{\pm}\phi_a + b_j^{\pm}[\phi_b + p_j\phi_a \log|\eta|] \, , \tag{3.7}$$

where

$$\phi_a = \eta + \tfrac{1}{2}p_j\eta^2 + \dots \, , \qquad \phi_b = 1 + q_j\eta^2 + \dots \, . \tag{3.8}$$

The $O(\epsilon^{\frac{6}{5}})$ term is introduced to take account of nonlinear interactions inside the critical layer, and has the form $v_2 = \bar{v}_2(Z, y, t_1)E^2 + \text{c.c.}$ The function $\bar{v}_2$ satisfies the same equation and hence has the same asymptotic behaviour as $\bar{v}_1$ provided $\alpha$ is replaced by $2\alpha$. Later, however, $\bar{v}_2$ will be found to be identically zero (see also Hickernell 1984).

The $O(\epsilon^{\frac{7}{5}})$ term $v_3$ has the form $v_3 = \bar{v}_3(Z, y, t_1)E + \text{c.c.}$ The function $\bar{v}_3$ satisfies the inhomogeneous Rayleigh equation

$$\left\{ \mathrm{D}^2 - \left( \alpha^2 + \frac{\bar{U}_{yy}}{\bar{U} - c} \right) \right\} \bar{v}_3$$
$$= (\mathrm{i}\alpha)^{-1} \left\{ \left[ -\frac{\partial A}{\partial t_1} - (\mathrm{i}\alpha \bar{U}_\tau \tau_1) A \right] \frac{\bar{U}_{yy}}{(\bar{U} - c)^2} + \frac{\mathrm{i}\alpha \bar{U}_{yy\tau} \tau_1 A}{\bar{U} - c} \right\} \bar{v}_1 - \frac{\partial^2 A}{\partial Z^2} \bar{v}_1 , \qquad (3.9)$$

where the last term of the above equation represents the three-dimensional distortion of a basically two-dimensional wave. As $\eta \to \pm 0$,

$$\bar{v}_3 \to -b_j^{\pm} r_j \log|\eta| + (a_j^{\pm} r_j + b_j^{\pm} s_j)\eta \log|\eta| + \dots$$
$$+ c_j^{\pm} \phi_a + d_j^{\pm} [\phi_b + p_j \phi_a \log|\eta|] , \qquad (3.10)$$

where

$$p_j = \frac{\bar{U}_{yy}}{\bar{U}_y} , \quad q_j = \tfrac{1}{2}\alpha^2 + \frac{1}{2} \frac{\bar{U}_{yyy}}{\bar{U}_y} - \frac{\bar{U}_{yy}^2}{\bar{U}_y^2} , \qquad (3.11)$$

$$r_j = (\mathrm{i}\alpha)^{-1} \frac{\bar{U}_{yy}}{\bar{U}_y^2} \left[ -\frac{\partial A}{\partial t_1} - (\mathrm{i}\alpha \bar{U}_\tau \tau_1) A \right] , \qquad (3.12)$$

$$s_j = (\mathrm{i}\alpha)^{-1} \left\{ (-\mathrm{i}\alpha \bar{U}_{y\tau} \tau_1 A) \frac{\bar{U}_{yy}}{\bar{U}_y^2} + (\mathrm{i}\alpha \tau_1 A) \frac{\bar{U}_{yy\tau}}{\bar{U}_y} \right.$$
$$\left. + \left( -\frac{\partial A}{\partial t_1} - \mathrm{i}\alpha \bar{U}_\tau \tau_1 A \right) \frac{\bar{U}_y \bar{U}_{yyy} - \bar{U}_{yy}^2}{\bar{U}_y^3} \right\} . \qquad (3.13)$$

Here all the basic-flow quantities are evaluated at time $\tau_0$ and the critical lever $y_c^j$. The jumps $(a_j^+ - a_j^-)$ etc. will be determined by analysing the critical layers.

The leading-order cross-flow component $w_1$ satisfies

$$\left( \frac{\partial}{\partial t} + \bar{U} \frac{\partial}{\partial x} \right) \frac{\partial w_1}{\partial y} + \bar{U}_y \frac{\partial w_1}{\partial x} = \left( \frac{\partial}{\partial t} + \bar{U}_y \frac{\partial}{\partial x} \right) \frac{\partial v_1}{\partial Z} . \qquad (3.14)$$

This suggests that $w_1 = A_Z(Z, t_1)\bar{w}_1(y)E + \text{c.c.}$ We find

$$\bar{w}_1 = \alpha^{-2} \left[ \bar{v}_{1,y} - \frac{\bar{U}_y}{\bar{U} - c} \bar{v}_1 \right] . \qquad (3.15)$$

The streamwise velocity $u_1$ has the form $u_1 = A(Z, t_1)\bar{u}_1(y)E + \text{c.c.}$ It follows from the continuity equation that

$$\bar{u}_1 = \mathrm{i}\alpha^{-1} \bar{v}_{1,y} . \qquad (3.16)$$

As for $v_2$, we find that $u_2$ at $O(\epsilon^{\frac{6}{5}})$ is identically zero. The solution for $u_3$ is obtained from the continuity equation as $u_3 = \bar{u}_3(y, Z, t_1)E + \text{c.c.}$, where

$$\bar{u}_3 = \mathrm{i}\alpha^{-1} \{ A(Z, t_1)\bar{v}_{3,y} + A_{ZZ}(Z, t_1)\bar{w}_1 \} . \qquad (3.17)$$

The pressure $p_1$ can be written as $p_1 = A(Z,t_1)\bar{p}_1(y)E + \text{c.c.}$, and

$$\bar{p}_1 = \mathrm{i}\alpha^{-1}[\bar{U}_y\bar{v}_1 - (\bar{U}-c)\bar{v}_{1,y}] \; . \tag{3.18}$$

We now introduce an inner variable:

$$Y = \frac{\eta}{\epsilon^{\frac{2}{3}}} = \frac{y - y_c^j}{\epsilon^{\frac{2}{3}}} \; . \tag{3.19}$$

The outer expansions written in terms of the inner variable are

$$\begin{aligned} v \sim\; & \epsilon b_j^{\pm}AE + \epsilon^{\frac{7}{3}}\log\epsilon^{\frac{2}{3}}(-b_j^{\pm}r_j + b_j^{\pm}p_jAY)E \\ & +\epsilon^{\frac{7}{3}}[(-b_j^{\pm}r_j\log|Y| + d_j^{\pm}) + A(a_j^{\pm}Y + b_j^{\pm}p_jY\log|Y|)]E \\ & +\epsilon^{\frac{9}{3}}\log\epsilon^{\frac{2}{3}}[(a_j^{\pm}r_j + b_j^{\pm}s_j + d_j^{\pm}p_j)Y + \frac{1}{2}Ap_jb_j^{\pm}Y^2]E \\ & +\epsilon^{\frac{9}{3}}\{c_j^{\pm}Y + (a_j^{\pm}r_j + b_j^{\pm}s_j + d_j^{\pm}p_j)Y\log|Y| + O(Y^2\log|Y|)\}E + \text{c.c.} + \dots \; , \end{aligned} \tag{3.20}$$

$$\begin{aligned} u \sim\; & \epsilon\log\epsilon^{\frac{2}{3}}\mathrm{i}\alpha^{-1}b_j^{\pm}p_jAE + \epsilon\mathrm{i}\alpha^{-1}\left\{b_j^{\pm}p_jA\log|Y| \right. \\ & \left. +A(a_j^{\pm} + b_j^{\pm}p_j) - b_j^{\pm}r_jY^{-1} - \alpha^{-2}b_j^{\pm}A_{ZZ}Y^{-1}\right\}E + \text{c.c.} + \dots \; , \end{aligned} \tag{3.21}$$

$$w \sim -\epsilon^{\frac{4}{3}}\alpha^{-2}b_j^{\pm}A_ZY^{-1}E + \text{c.c.} + \dots \; , \tag{3.22}$$

$$p \sim \epsilon\mathrm{i}\alpha^{-1}\bar{U}_yb_j^{\pm}E + \text{c.c.} + \dots \; . \tag{3.23}$$

### 3.2. *Solvability condition*

Multiplying both sides of (3.9) by $\bar{v}_1$ and integrating from 0 to $+\infty$, we obtain the solvability condition for (3.9):

$$-\sum_j \mathrm{FP}(\triangle_j) = \mathrm{i}\alpha^{-1}J_1\frac{\partial A}{\partial t_1} + J_2\tau_1A - J_3\frac{\partial^2A}{\partial Z^2} \; , \tag{3.24}$$

where the sum is over all critical layers, and FP denotes the finite part of $\triangle_j$:

$$\triangle_j = (\bar{v}_{3y}\bar{v}_1 - \bar{v}_3\bar{v}_{1y})|_{y\to y_c^{j+}} - (\bar{v}_{3y}\bar{v}_1 - \bar{v}_3\bar{v}_{1y})|_{y\to y_c^{j-}} \; . \tag{3.25}$$

The constants $J_1$, $J_2$ and $J_3$ are defined as follows:

$$J_1 = \int_0^{+\infty}\frac{\bar{U}_{yy}}{(\bar{U}-c)^2}\bar{v}_1^2\mathrm{d}y \; , \tag{3.26}$$

$$J_2 = \int_0^{+\infty}\left[-\frac{\bar{U}_{yy}\bar{U}_\tau}{(\bar{U}-c)^2} + \frac{\bar{U}_{yy\tau}}{(\bar{U}-c)}\right]\bar{v}_1^2\mathrm{d}y \; , \quad J_3 = \int_0^{\infty}\bar{v}_1^2(y)\mathrm{d}y \; . \tag{3.27}$$

Note that $J_1$ and $J_2$ are singular and should be interpreted in the sense of Hadamard. Substituting (3.7), (3.10) together with (3.8) into (3.25), we find

$$\triangle_j = b_j(c_j^+ - c_j^-) - b_jr_j(a_j^+ - a_j^-) - b_jp_j(d_j^+ - d_j^-) - (a_j^+d_j^+ - a_j^-d_j^-) \; . \tag{3.28}$$

The amplitude equation will follow from (3.24) and (3.28) after $(a_j^+ - a_j^-)$ etc. are determined. Here it is worth noting that nonlinearity influences the evolution of the disturbance by altering these jumps; thus in the analysis of the critical layers, it is sufficient to solve for those parts of the solutions contributing to these jumps.

## 4. Inner expansion

Equations (3.20)–(3.23) suggest that the expansions within the $j$th critical layer take the form

$$u = \epsilon U_1 + \epsilon^{\frac{6}{5}} U_2 + \epsilon^{\frac{7}{5}} U_3 + \dots\ , \tag{4.1}$$

$$v = \epsilon V_0 + \epsilon^{\frac{6}{5}} V_{01} + \epsilon^{\frac{7}{5}} V_1 + \epsilon^{\frac{8}{5}} V_2 + \epsilon^{\frac{9}{5}} V_3 + \dots\ , \tag{4.2}$$

$$w = \epsilon^{\frac{4}{5}} W_1 + \epsilon W_2 + \epsilon^{\frac{6}{5}} W_3 + \dots\ , \tag{4.3}$$

$$p = \epsilon P_1 + \epsilon^{\frac{6}{5}} P_2 + \epsilon^{\frac{7}{5}} P_3 + \dots\ , \tag{4.4}$$

where $O(\epsilon^n \log \epsilon^{\frac{1}{5}})$ terms have not been explicitly included. This is because as far as deriving the amplitude equation is concerned, they are passive in the sense that matches at these orders are guaranteed once the solutions at $O(\epsilon^n)$ match.

The leading-order term $V_0$ satisfies

$$\mathrm{L}_0 V_{0,YY} = 0\ ,$$

where

$$\mathrm{L}_0 = \left[\frac{\partial}{\partial t_1} + (\bar{U}_y Y + \bar{U}_\tau \tau_1)\frac{\partial}{\partial x}\right] - \lambda \frac{\partial^2}{\partial Y^2}\ . \tag{4.5}$$

The solution which matches the outer expansion is

$$V_0 = \hat{A}(Z, t_1)E + \text{c.c.}\ , \tag{4.6}$$

where $\hat{A} = b_j A$ and $b_j^+ = b_j^- = b_j$, i.e. the jump $(b_j^+ - b_j^-)$ is zero.

The function $V_{01}$ satisfies the same equation as $V_0$. The appropriate solution is

$$V_{01} = \hat{B}(Z, t_1) + \hat{C}(Z, t_1)E^2 + \text{c.c.}$$

The expansion of the $y$-momentum equation gives

$$P_1 = \mathrm{i}\alpha^{-1}\bar{U}_y \hat{A}(Z, t_1)E + \text{c.c.}$$

The cross-flow velocity $W_1$ has the solution $W_1 = \hat{W}_1(Y, Z, t_1)E + \text{c.c.}$, where $\hat{W}_1$ is governed by

$$\left\{\frac{\partial}{\partial t_1} + \mathrm{i}\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1) - \lambda \frac{\partial^2}{\partial Y^2}\right\} \hat{W}_1 = -\mathrm{i}\alpha^{-1}\bar{U}_y A_Z(Z, t_1)\ . \tag{4.7}$$

We solve (4.7) using the Fourier transform method, and obtain

$$\hat{W}_1 = -\mathrm{i}\alpha^{-1}\bar{U}_y \int_0^{+\infty} \hat{A}_Z(Z, t_1 - \xi)\mathrm{e}^{-s\xi^3 - \mathrm{i}\Omega\xi}\mathrm{d}\xi\ , \tag{4.8}$$

where

$$s = \tfrac{1}{3}\lambda\alpha^2\bar{U}_y^2\ , \quad \text{and} \quad \Omega = \alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)\ . \tag{4.9}$$

The vertical velocity component $V_1$ satisfies

$$\mathrm{L}_0 V_{1,YY} = \mathrm{L}_1 V_0\ , \tag{4.10}$$

where

$$\mathrm{L}_1 = -(\tfrac{1}{2}\bar{U}_{yy}Y^2 + \bar{U}_{y\tau}\tau_1 Y + \tfrac{1}{2}\bar{U}_{\tau\tau}\tau_1^2)\frac{\partial^3}{\partial x \partial Y^2} + \bar{U}_{yy}\frac{\partial}{\partial x}\ . \tag{4.11}$$

Let $V_1 = \hat{V}_1(Y, Z, t_1)E + \text{c.c.}$; then we find

$$\hat{V}_{1,YY} = \mathrm{i}\alpha\bar{U}_{yy} \int_0^{+\infty} \hat{A}(Z, t_1 - \xi)\mathrm{e}^{-s\xi^3 - \mathrm{i}\Omega\xi}\mathrm{d}\xi\ . \tag{4.12}$$

We note that $\hat{V}_1$ is equivalent to $\Phi_3$ in the purely two-dimensional case (Wu & Cowley 1993). By analogy, we obtain the jump conditions

$$a_j^+ - a_j^- = \pi \mathrm{i} p_j b_j \mathrm{Sgn}(\bar{U}_y)\ , \quad d_j^+ - d_j^- = -\pi \mathrm{i} r_j b_j \mathrm{Sgn}(\bar{U}_y)\ . \tag{4.13}$$

Similarly, we write $U_1 = \hat{U}_1(Y, Z, t_1)E + \text{c.c.}$; then it follows from the continuity equation that

$$\hat{U}_{1,Y} = -\mathrm{i}\alpha^{-1}\bar{U}_y^2 \int_0^{+\infty} \xi \hat{A}_{ZZ}(Z, t_1 - \xi) \mathrm{e}^{-s\xi^3 - \mathrm{i}\Omega\xi} \mathrm{d}\xi$$
$$-\bar{U}_{yy} \int_0^{+\infty} \hat{A}(Z, t_1 - \xi) \mathrm{e}^{-s\xi^3 - \mathrm{i}\Omega\xi} \mathrm{d}\xi\ . \tag{4.14}$$

The function $V_2$ satisfies

$$\mathrm{L}_0 V_2 = L_1 V_{01} + \frac{\partial^2}{\partial Y^2}\left\{\frac{\partial}{\partial x} V_0 U_1 + \frac{\partial}{\partial Z} V_0 W_1\right\} + \dots\ . \tag{4.15}$$

The first forcing term above is analogous to that in (4.10) and thus produces an analogous velocity jump, while the solution driven by the second term produces no velocity jump. Because $(2\alpha, c)$ is not a fundamental mode of Rayleigh's equation (Cowley 1987), $v_2$, and hence $u_2$ and $V_{01}$ should be identically zero.

The function $W_2$ satisfies the equation

$$\mathrm{L}_0 W_2 = -\frac{\partial}{\partial Y}(V_0 W_1)\ . \tag{4.16}$$

The solution can be written as

$$W_2 = \hat{W}_2^{(0)} + \hat{W}_2^{(2)} E^2 + \text{c.c.} \tag{4.17}$$

It is found that at the next order the interaction between the fundamental and the harmonic $\hat{W}_2^{(2)}$ does *not* contribute to the jump $(c_j^+ - c_j^-)$, and hence does not affect the amplitude equation. Therefore, we solve for the cross-flow distortion $\hat{W}_2^{(0)}$ only, and obtain

$$\hat{W}_2^{(0)} = \bar{U}_y^2 \int_0^{+\infty}\int_0^{+\infty} \xi \hat{A}^*(Z, t_1 - \xi)\hat{A}_Z(Z, t_1 - \xi - \zeta) K_0(\xi, \zeta) \mathrm{e}^{-\mathrm{i}\Omega\xi} \mathrm{d}\xi \mathrm{d}\zeta\ , \tag{4.18}$$

where $*$ represents the complex conjugate, and

$$K_0(\xi, \zeta) = \mathrm{e}^{-s(\xi^3 + 3\xi^2\zeta)}\ .$$

We now turn to solve for $U_2$, which is governed by

$$\mathrm{L}_0 U_{2,Y} = \bar{U}_y \frac{\partial W_2}{\partial Z} - \frac{\partial^2}{\partial Y^2}(V_0 U_1)\ . \tag{4.19}$$

The solution takes the form

$$U_2 = \hat{U}_2^{(0)} + \hat{U}_2^{(2)} E^2 + \text{c.c.} \tag{4.20}$$

As for $W_2$, it sufficient to solve for the mean-flow distortion $\hat{U}_2^{(0)}$ only. We find

$$\hat{U}_{2,Y}^{(0)} = -\mathrm{i}\alpha \bar{U}_{yy}\bar{U}_y \int_0^{+\infty}\int_0^{+\infty} \xi \hat{A}^*(Z, t_1 - \zeta)\hat{A}(Z, t_1 - \xi - \zeta) K_0(\xi, \zeta) \mathrm{e}^{-\mathrm{i}\Omega\xi} \mathrm{d}\xi \mathrm{d}\zeta$$
$$+\bar{U}_y^3 \int_0^{+\infty}\int_0^{+\infty} \xi^2 \hat{A}^*(Z, t_1 - \zeta)\hat{A}_{ZZ}(Z, t_1 - \xi - \zeta) K_0(\xi, \zeta) \mathrm{e}^{-\mathrm{i}\Omega\xi} \mathrm{d}\xi \mathrm{d}\zeta$$

$$+\bar{U}_y^3\int_0^{+\infty}\int_0^{+\infty}\xi\zeta[\hat{A}^*(Z,t_1-\zeta)\hat{A}_Z(Z,t_1-\xi-\zeta)]_Z K_0(\xi,\zeta)\mathrm{e}^{-\mathrm{i}\Omega\xi}\mathrm{d}\xi\mathrm{d}\zeta\ . \quad (4.21)$$

To determine the jump $(c_j^+ - c_j^-)$, we need to solve for $V_3$. This term satisfies

$$\mathrm{L}_0 V_3 = \mathrm{L}_1 V_1 + \mathrm{L}_2 V_0 + \mathrm{i}\alpha\hat{A}\hat{U}^{*(0)}_{2,YY}E + \frac{\partial}{\partial Z}(\hat{A}\hat{W}^{*(0)}_{2,YY})E + \text{c.c.} + \dots\ , \quad (4.22)$$

where the forcing terms which do not contribute to the jump are omitted, and

$$\mathrm{L}_2 = -[\tfrac{1}{6}\bar{U}_{yyy}Y^3 + \bar{U}_{yy\tau}\tau_1 Y^2 + \bar{U}_{y\tau\tau}\tau_1^2 Y + \tfrac{1}{6}\bar{U}_{\tau\tau\tau}\tau_1^3]\frac{\partial^3}{\partial x\partial Y^2}$$
$$+[\bar{U}_{yyy}Y + \bar{U}_{yy\tau}\tau_1]\frac{\partial}{\partial x} - \left[\frac{\partial}{\partial t_1} + (\bar{U}_y Y + \bar{U}_\tau\tau_1)\frac{\partial}{\partial x}\right]\frac{\partial^2}{\partial x^2}\ . \quad (4.23)$$

We first seek the solution driven by the linear forcing term, i.e. $(\mathrm{L}_1 V_1 + \mathrm{L}_2 V_0)$, which we denote by $F^{(l)}(Z,Y,t_1)E$. We note that $F^{(l)}(Z,Y,t_1)$ is the same as $F^{(l)}(Y,t_1)$ of the two-dimensional case (Wu & Cowley 1993) except that the former depends on $Z$. Let $\hat{V}_3^{(l)}E$ denote the solution driven by $F^{(l)}(Z,Y,t_1)E$; then by a similar procedure, we find that as $Y \to \pm\infty$

$$\hat{V}^{(l)}_{3,Y} \to (a_j^+ p_j + 2q_j b_j + \tfrac{1}{2}p_j^2 b_j)Y + (a_j^+ r_j + p_j d_j^+ + s_j b_j)\log|Y|$$
$$+\{\pm\tfrac{1}{2}\pi\mathrm{i}\,\mathrm{Sgn}(\bar{U}_y)(a_j^+ r_j + p_j d_j^+ + s_j b_j) + \dots\} + O(Y^{-1})\ . \quad (4.24)$$

Let $\hat{V}_3^{(n)}E$ denotes the solution driven by the nonlinear forcing term, i.e. the third and fourth terms in (4.22); harmonics can be ignored as far as deriving the amplitude equation is concerned. It is found that

$$\hat{V}^{(n)}_{3,YY} = -\mathrm{i}\alpha^3\bar{U}_y^2\bar{U}_{yy}\Pi_1 - \alpha^2\bar{U}_y^4\Pi_2\ , \quad (4.25)$$

where

$$\Pi_1 = \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\xi^2\hat{A}(Z,t_1-\zeta)\hat{A}(Z,t_1-\zeta-\eta)\hat{A}^*(Z,t_1-\xi-\zeta-\eta)$$
$$\times K_1(\xi,\eta,\zeta)\mathrm{d}\xi\mathrm{d}\zeta\mathrm{d}\eta\ , \quad (4.26)$$

$$\Pi_2 = \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}K_1(\xi,\eta,\zeta)\Big\{\xi^3\hat{A}(Z,t_1-\zeta)\hat{A}(Z,t_1-\zeta-\eta)\hat{A}^*_{ZZ}(Z,t_1-\xi-\zeta-\eta)$$
$$+\xi^2\eta\hat{A}(Z,t_1-\zeta)[\hat{A}(Z,t_1-\zeta-\eta)\hat{A}^*_Z(Z,t_1-\xi-\zeta-\eta)]_Z$$
$$+\xi^3[\hat{A}(Z,t_1-\zeta)\hat{A}(Z,t_1-\zeta-\eta)\hat{A}^*_Z(Z,t_1-\xi-\zeta-\eta)]_Z\Big\}\mathrm{d}\xi\mathrm{d}\zeta\mathrm{d}\eta\ . \quad (4.27)$$

The kernel $K_1(\xi,\eta,\zeta)$ is

$$K_1(\xi,\eta,\zeta) = \mathrm{e}^{-s[2\xi^2+3\xi^2\eta-(\zeta-\xi)^3]-\mathrm{i}\Omega(\zeta-\xi)}\ .$$

Using (4.24)–(4.27), and matching $(\hat{V}^{(l)}_{3,Y} + \hat{V}^{(n)}_{3,Y})$ with the outer expansion, we obtain the final jump condition

$$c_j^+ - c_j^-$$
$$= \pi\mathrm{i}\,\mathrm{sgn}(\bar{U}_y)(a_j^+ r_j + p_j d_j^+ + s_j b_j)$$
$$-2\pi\mathrm{i}\alpha^2\bar{U}_{yy}|\bar{U}_y|\int_0^{+\infty}\int_0^{+\infty}\xi^2\hat{A}(Z,t_1-\xi)\hat{A}(Z,t_1-\xi-\eta)\hat{A}^*(Z,t_1-2\xi-\eta)K_j(\xi,\eta|\lambda)\mathrm{d}\xi\mathrm{d}\eta$$
$$-2\pi\alpha|\bar{U}_y|^3\int_0^{+\infty}\int_0^{+\infty}\xi^3\hat{A}(Z,t_1-\xi)\hat{A}(Z,t_1-\xi-\eta)\hat{A}^*_{ZZ}(Z,t_1-2\xi-\eta)K_j(\xi,\eta|\lambda)\mathrm{d}\xi\mathrm{d}\eta$$

$$-2\pi\alpha|\bar{U}_y|^3\int_0^{+\infty}\int_0^{+\infty}\xi^2\eta\hat{A}(Z,t_1-\xi)[\hat{A}(Z,t_1-\xi-\eta)\hat{A}^*_Z(Z,t_1-2\xi-\eta)]_Z K_j(\xi,\eta|\lambda)\mathrm{d}\xi\mathrm{d}\eta$$
$$-2\pi\alpha|\bar{U}_y|^3\int_0^{+\infty}\int_0^{+\infty}\xi^3[\hat{A}(Z,t_1-\xi)\hat{A}(Z,t_1-\xi-\eta)\hat{A}^*_Z(Z,t_1-2\xi-\eta)]_Z K_j(\xi,\eta|\lambda)\mathrm{d}\xi\mathrm{d}\eta, \tag{4.28}$$

where

$$K_j(\xi,\eta|\lambda)=\mathrm{e}^{-s(2\xi^3+3\xi^2\eta)}. \tag{4.29}$$

Here the suffix $j$ refers to the $j$th critical layer, and the argument $\lambda$ refers to the dependence on $\lambda$ through (4.9). It is worth noting that while both $(a_j^+-a_j^-)$ and $(d_j^+-d_j^-)$ correspond to the classic $\pm\pi$ phase shift at the logarithmic branch point, $(c_j^+-c_j^-)$ is modified by nonlinearity. This modification has a significant effect on the evolution of the disturbance. Also note that nonlinear effects come into play by producing spanwise-dependent mean-flow distortions, i.e. $\hat{U}_2^{(0)}$ and $\hat{W}_2^{(0)}$, while the harmonics have no significance. Though this resembles the wave–vortex interaction, it is *not* appropriate at this stage to link it to the wave–vortex interaction theory of Smith & Walton (1989) and Hall & Smith (1990). The reasons are as follows. Firstly, this feature is also exhibited in the purely two-dimensional case (Wu & Cowley 1993). Secondly, in our analysis the harmonics are passive because of the particular structure of the solution rather than because they have a smaller magnitude. Indeed, in our study they are of the same magnitude as the mean-flow distortions (see (4.17) and (4.20)), while in the wave-vortex interaction theories mentioned above, the latter usually have a much larger magnitude than the former. Nevertheless, as in Wu *et al.* (1993), the viscous limit of our approach appears to be linked with the wave–vortex interaction approach (see §6).

## 5. Amplitude equation

Inserting the jumps (4.13), and (4.28) into (3.28), and using (3.24), we obtain the crucial amplitude equation as follows:

$$\frac{\partial A}{\partial t_1}-q\frac{\partial^2 A}{\partial Z^2}$$
$$=g_0\tau_1 A+\int_0^{+\infty}\int_0^{+\infty}\sum g_jK_j(\xi,\eta|\lambda)\xi^2A(Z,t_1-\xi)A(Z,t_1-\xi-\eta)$$
$$\times A^*(Z,t_1-2\xi-\eta)\mathrm{d}\xi\mathrm{d}\eta$$
$$+\int_0^{+\infty}\int_0^{+\infty}\sum h_jK_j(\xi,\eta|\lambda)\xi^3A(Z,t_1-\xi)A(Z,t_1-\xi-\eta)A^*_{ZZ}(Z,t_1-2\xi-\eta)\mathrm{d}\xi\mathrm{d}\eta$$
$$+\int_0^{+\infty}\int_0^{+\infty}\sum h_jK_j(\xi,\eta|\lambda)\xi^2\eta A(Z,t_1-\xi)[A(Z,t_1-\xi-\eta)A^*_Z(Z,t_1-2\xi-\eta)]_Z\mathrm{d}\xi\mathrm{d}\eta$$
$$+\int_0^{+\infty}\int_0^{+\infty}\sum h_jK_j(\xi,\eta|\lambda)\xi^3[A(Z,t_1\xi)A(Z,t_1-\xi-\eta)A^*_Z(Z,t_1-2\xi-\eta)]_Z\mathrm{d}\xi\mathrm{d}\eta, \tag{5.1}$$

where the sums are over all critical layers; for the Stokes layer under consideration, there are two. The kernel $K_j(\xi,\eta|\lambda)$ represents the effect of viscosity, and is defined by (4.29). For the inviscid case, $\lambda=0$; so $K_j(\xi,\zeta)=1$. The coefficients involved are expressed in terms of the basic-flow profile and the eigenfunction, namely

$$g_0 = f_0/f \ , \quad g_j = 2\pi i\alpha^2 b_j^2 |b_j|^2 \bar{U}_{yy} |\bar{U}_y| / f \ , \tag{5.2}$$

$$q = J_3/f \ , \quad h_j = 2\pi\alpha b_j^2 |b_j|^2 |\bar{U}_y|^3 / f \ , \tag{5.3}$$

$$f = i\alpha^{-1} \left\{ \sum_j \pi b_j \left[ 2i \frac{\bar{U}_{yy}}{|\bar{U}_y|^2} a_j^+ + i b_j \frac{\bar{U}_y \bar{U}_{yyy} - \bar{U}_{yy}^2}{\bar{U}_y |\bar{U}_y|^2} + b_j \pi \frac{\bar{U}_{yy}^2}{\bar{U}_y^3} \right] + J_1 \right\}, \tag{5.4}$$

$$f_0 = \sum_j \left\{ 2\pi i b_j \frac{\bar{U}_{yy} \bar{U}_\tau}{|\bar{U}_y|^2} a_j^+ - \pi i b_j^2 \left[ \frac{\bar{U}_{yy\tau}}{|\bar{U}_y|} - \frac{\bar{U}_{yy} \bar{U}_{y\tau}}{|\bar{U}_y|^2} - \frac{(\bar{U}_y \bar{U}_{yyy} - \bar{U}_{yy}^2) \bar{U}_\tau}{\bar{U}_y |\bar{U}_y|^2} \right] \right.$$
$$\left. + \pi^2 b_j^2 \frac{\bar{U}_{yy} \bar{U}_\tau}{\bar{U}_y^3} \right\} - J_2 \ . \tag{5.5}$$

They can be evaluated in the same way as described by Wu & Cowley (1993).

To the best of our knowledge, the modulation equation (5.1) has not been derived before. We note that the first nonlinear term is associated with the logarithmic branch-point singularity. Since it is the same as that in the amplitude equation for a purely two-dimensional wave (Hickernell 1984; Wu & Cowley 1993), we shall refer to it as the 'two-dimensional nonlinear term'. The rest of the nonlinear terms are associated with the pole type of singularity and hence shall be referred to as 'three-dimensional nonlinear terms' . A notable feature is that they contain spatial derivatives, including the highest derivative.

As we have emphasized, the scaling of this analysis also applies to flows with regular critical layers, e.g. free shear layers. The major difference is that for these flows we have to consider (streamwise) spatial rather than temporal development. Since the formation of streamwise vortices in these flows is of great importance and has attracted much attention (see §1), in Appendix A we modify our analysis to that case. Thereby we derive the appropriate amplitude equation together with the expressions for the coefficients. As we can see, a similar amplitude equation holds, except that the 'two-dimensional nonlinear term' disappears and a spatial variable $x_1$ describing the streamwise evolution plays the role of time $t_1$; see (A 11). Therefore it is sufficient for us to discuss the property of (5.1) only, while treating free shear layers as a special case with $g_j$ being zero. However, later on our numerical study will concentrate on free shear layers. We note that the amplitude equation (A 11) also applies to the boundary layer with a weak adverse pressure gradient which is inviscidly unstable to long-wavelength Rayleigh modes.

A property of the modulation equation (5.1) is that if

$$A = B(t) e^{i\beta Z} \ , \tag{5.6}$$

then the 'three-dimensional nonlinear terms' are identically zero, and as a result $B$ satisfies the same equation as the amplitude of a purely two-dimensional wave (Hickernell 1984; Wu & Cowley 1993). Note that (5.6) corresponds to the case where the disturbance is a *single* oblique wave. Thus the property above helps to explain why the pole type of singularity has no significance for the development of a single oblique wave (Goldstein & Leib 1989).

In this study, we obtain a single amplitude equation, while in the case of Davey *et al.* (1974), the amplitude equation was coupled with an additional function related to the secular pressure. However, we note that if it were assumed in their analysis, as it is in our case, that the amplitude depended on the spanwise coordinate only, then $B = |A|^2$ would immediately follow from their (2.28); in this case their coupled equations (2.31) and (2.34) are reduced to a single equation, with an additional nonlinear term proportional to $qA|A|^2$ representing the three-dimensional nonlinear effect.

So far we have implicitly assumed that the amplitude function $A(Z, t_1)$ is defined from $-\infty$. If we assume that the disturbance is introduced at a specific time, say at $t_1 = 0$, as in the study of Hickernell (1984), then the integral limit in (5.1) becomes $\int_0^{\frac{t_1}{2}} \int_0^{t_1-2\xi}$, identical to that of Hickernell. Nevertheless, the amplitude equation can still be written as (5.1) provided that $A(Z, t_1)$ is taken to be zero when $t_1 < 0$.

Following the spirit of Stewartson & Stuart (1971), we require the solution to (5.1) to match onto the initial linear evolution stage. A typical case is that the disturbance is initially centered at spanwise location, say, $Z = 0$. Then as in the studies of Hocking *et al.* (1972), and Hocking & Stewartson (1972), we have

$$A \to \frac{\Delta}{t_1^{\frac{1}{2}}} \exp\left(-\frac{Z^2}{4qt_1}\right) \qquad \text{as} \qquad t_1 \to 0\,, \tag{5.7}$$

where $\Delta$ is a measure of the 'strength' of the disturbance. Equations (5.1) and (5.7) thus describe the evolution of the disturbance. Note that the solution is required to match as $t_1 \to 0$ rather than as $t_1 \to -\infty$ (cf. Goldstein & Leib 1989). This is because the problem of a localized disturbance undergoing a diffusion from the 'infinite past' is ill-posed; hence the time origin must be specified.

In addition to the localized initial condition, another important case is where the initial disturbance is periodic in the spanwise direction, e.g.

$$A \to [\breve{a}_0 + \breve{a}_1 \cos\beta Z e^{-q\beta^2 t_1}] e^{g_0\tau_1 t_1} \quad \text{as} \quad t_1 \to -\infty\,.$$

Here we require $\mathrm{Re}(g_0\tau_1 - q\beta^2) > 0$, which imposes a restriction on $\tau_1$ and $\beta$. The periodic initial distribution appears to be relevant to the experiments of Nygaard & Glezer (1991) and Lasheras & Choi (1988). However, we shall leave the numerical investigation of this case to the future because a spectral method, which is entirely different from the finite-difference method adopted for a localized disturbance, must be employed.

For convenience, we rescale the amplitude equation and the initial condition (5.1) and (5.7) by introducing new variables (cf. Goldstein & Choi 1989), namely

$$\bar{A} = Ae^{-\mathrm{i}g_{0i}\tau_1 t_1}|h|^{\frac{1}{2}}|q_r|^{-\frac{1}{2}}/(g_{0r}\tau_1)^{\frac{5}{2}}\,, \quad \bar{A}_0 = A_0|h|^{\frac{1}{2}}|q_r|^{-\frac{1}{2}}/(g_{0r}\tau_1)^{\frac{5}{2}}\,, \tag{5.8}$$

$$\bar{t} = g_{0r}\tau_1 t_1\,, \quad \bar{Z} = (g_{0r}\tau_1)^{\frac{1}{2}}|q_r|^{-\frac{1}{2}}Z\,, \quad \bar{\lambda} = \lambda/(g_{0r}\tau_1)^3\,, \tag{5.9}$$

where $h = \sum h_j$, $g_{0r}$ and $g_{0i}$ are the real and imaginary parts of $g_0$ respectively, and $q_r$ is the real part of $q$. The evolution equation and the initial condition then become

$$\begin{aligned}
&\frac{\partial \bar{A}}{\partial \bar{t}} - (1 + \mathrm{i}\bar{q})\frac{\partial^2 \bar{A}}{\partial \bar{Z}^2} \\
&= \bar{A} + \hat{s}_0 \int_0^{+\infty}\int_0^{+\infty} \sum \bar{g}_j K_j(\xi,\eta|\lambda)\xi^2 \bar{A}(\bar{Z},\bar{t}-\xi)\bar{A}(\bar{Z},\bar{t}-\xi-\eta)\bar{A}^*(\bar{Z},\bar{t}-2\xi-\eta)\mathrm{d}\xi\mathrm{d}\eta \\
&\quad + \int_0^{+\infty}\int_0^{+\infty} \sum \bar{h}_j K_j(\xi,\eta|\lambda)\,\{\xi^3 \bar{A}(\bar{Z},\bar{t}-\xi)\bar{A}(\bar{Z},\bar{t}-\xi-\eta)A^*_{\bar{Z}\bar{Z}}(\bar{Z},\bar{t}_1-2\xi-\eta) \\
&\quad + \xi^2\eta \bar{A}(\bar{Z},\bar{t}-\xi)[\bar{A}(\bar{Z},\bar{t}-\xi-\eta)\bar{A}^*_{\bar{Z}}(\bar{Z},\bar{t}-2\xi-\eta)]_{\bar{Z}} \\
&\quad + \xi^3\,[\bar{A}(\bar{Z},\bar{t}-\xi)\bar{A}(\bar{Z},\bar{t}-\xi-\eta)\bar{A}^*_{\bar{Z}}(\bar{Z},\bar{t}-2\xi-\eta)]_{\bar{Z}}\}\,\mathrm{d}\xi\mathrm{d}\eta\,,
\end{aligned} \tag{5.10}$$

$$\bar{A} \to \bar{A}_0(\bar{Z}, \bar{t})e^{\bar{t}} \quad \text{as} \quad \bar{t} \to 0, \tag{5.11}$$

where we have written $\bar{\lambda}$ as $\lambda$, $\bar{q} = q_i/q_r$, and $\bar{g}_j$, $\bar{h}_j$ are constants with unit modulus:

$$\bar{g}_j = g_j/|\sum g_j|, \quad \bar{h}_j = h_j/|\sum h_j|.$$

The real constant $\hat{s}_0$ is

$$\hat{s}_0 = |p_r||\sum g_j|/|\sum h_j|,$$

and is zero for regular critical layers.

## 6. Studies of the amplitude equation

### 6.1. *Very weak three-dimensionality limit*

The amplitude equation (5.1) is formally valid for $\beta \sim O(\epsilon^{\frac{1}{3}})$. We now consider the limit $\beta \ll O(\epsilon^{\frac{1}{3}})$, i.e. three-dimensionality is weaker. To this end, we introduce $\breve{Z} = \breve{\beta}Z$ with $\breve{\beta} \ll 1$. For singular critical layers, substituting $\breve{Z}$ into (5.1) and taking the limit $\breve{\beta} \to 0$, we find that $\partial^2 A/\partial Z^2$ and the 'three-dimensional nonlinear terms' can be dropped, leaving the 'two-dimensional nonlinear term' dominant, i.e. the disturbance can be treated effectively as purely two-dimensional (Hickernell 1984; Wu & Cowley 1993).

For regular critical layers ($\hat{s}_0 = 0$), we introduce $\breve{A} = \breve{\beta}A$ with $\breve{A} = O(1)$. Substituting this along with $\breve{Z}$ into (5.1), we find that $\breve{A}$ is governed, to leading order, by

$$\begin{aligned}\frac{\partial \breve{A}}{\partial t_1} = g_0\tau_1 A + \int_0^{+\infty}\int_0^{+\infty} \sum h_j K_j(\xi, \eta|\lambda) \Big\{ &\xi^3 \breve{A}(Z, t_1-\xi)\breve{A}(Z, t_1-\xi-\eta)\breve{A}^*_{ZZ}(Z, t_1-2\xi-\eta) \\ &+ \xi^2\eta \breve{A}(Z, t_1-\xi)[\breve{A}(Z, t_1-\xi-\eta)\breve{A}^*_Z(Z, t_1-2\xi-\eta)]_Z \\ &+ \xi^3[\breve{A}(Z, t_1-\xi)\breve{A}(Z, t_1-\xi-\eta)\breve{A}^*_Z(Z, t_1-2\xi-\eta)]_Z \Big\} d\xi d\eta, \end{aligned} \tag{6.1}$$

where we have dropped the breve over $Z$. The (unscaled) magnitude now is $O(\epsilon\breve{\beta}^{-1}) \equiv \bar{\epsilon}$, while the growth rate remains $O(\epsilon^{\frac{2}{3}})$. Equation (6.1) describes the evolution of a disturbance with very weak three-dimensionality. However it breaks down if three-dimensionality is too weak, i.e. if $\breve{\beta}$ is sufficiently small. More precisely, when the growth rate is of order of the square root of the (unscaled) amplitude, i.e. $\epsilon^{\frac{2}{3}} \sim (\epsilon\breve{\beta}^{-1})^{\frac{1}{2}}$, or $\breve{\beta} = O(\epsilon^{\frac{1}{3}})$, the critical layer becomes strongly nonlinear as for a purely two-dimensional disturbance (Goldstein & Leib 1988; Goldstein & Hultgren 1989), but subject to some modification due to the coupling with the three-dimensional velocity components. Such a problem is currently being investigated by Professor A.M. Messister (private communication), who independently obtained the scaling. In terms of the unscaled amplitude $\bar{\epsilon}$, the growth rate, the critical-layer width and the spanwise scale $(\beta\breve{\beta})$ are all of order $\bar{\epsilon}^{\frac{1}{2}}$. This crucial spanwise scale also follows directly from (2.7) by setting $\mu = O(\epsilon^{\frac{1}{2}})$. So we conclude that (5.1) is valid when $\epsilon^{\frac{1}{2}} \ll \beta \leqslant \epsilon^{\frac{1}{3}}$ while its limit form (6.1) is valid when $\epsilon^{\frac{1}{2}} \ll \beta \ll \epsilon^{\frac{1}{3}}$. Once $\beta = O(\epsilon^{\frac{1}{2}})$ a strongly nonlinear critical layer must be set up.

### 6.2. *Very viscous limit and relation to wave/vortex interaction approach*

The amplitude equation (5.1) is formally derived for $\epsilon = O(R^{-\frac{5}{6}})$. Next we turn to examine the very viscous limit $\epsilon \ll R^{-\frac{5}{6}}$, i.e. $\lambda \to +\infty$. This corresponds to the situation where the disturbance is relatively small so that the critical layers become viscous-dominated before nonlinearity comes into the reckoning. In Appendix B, it is shown that in this limit, equation (5.1) reduces to

$$\lambda^{-\frac{1}{3}}\frac{\partial A}{\partial \bar{t}_1} - q\frac{\partial^2 A}{\partial Z^2} = g_0\tau_1 A + \lambda^{-\frac{4}{3}} I_0|A|^2 A$$
$$+\lambda^{-\frac{4}{3}} J_0 A\int_0^{+\infty}\eta^{-\frac{1}{2}}[A(\bar{t}_1-\eta)A_Z^*(\bar{t}_1-\eta)]_Z\,\mathrm{d}\eta + O(\lambda^{-\frac{5}{3}}) , \qquad (6.2)$$

where $\bar{t}_1$ is defined in (B 1). The complex constants $I_0$ and $J_0$ are

$$I_0 = \frac{\hat{s}_0}{3^{\frac{2}{3}}\times 2^{\frac{1}{3}}}\Gamma\left(\tfrac{1}{3}\right)\sum g_j\beta_j^{-\frac{4}{3}} , \quad J_0 = \frac{\sqrt{3}}{36}\Gamma\left(\tfrac{1}{2}\right)\sum h_j\beta_j^{-\frac{3}{2}} ,$$

where the sums are again over all critical layers. Interestingly, in the viscous limit the nonlinear term is still non-local. Similar behaviour was observed by Wu *et al.* (1993). For (6.2) to be valid, we require $\lambda \gg R^{-\frac{1}{4}}$. This is because as $\lambda = O(R^{-\frac{1}{4}})$, $\tau_1$ ceases to be a parameter. Instead, it merges with $t_1$ and becomes a variable. Also the viscous sublayer adjacent to the wall will contribute an additional term proportion to $A$. For details of reasoning, see Wu *et al.* (1993).

For the regular critical layers ($I_0 = 0$), we rescale the equation by the substitutions:

$$Z = \lambda^{\frac{1}{6}}\bar{Z} , \quad \tau_1 = \lambda^{-\frac{1}{3}}\bar{\tau}_1, \quad A = \lambda^{\frac{2}{3}}\bar{A} ; \qquad (6.3)$$

the rescaled form of (6.2), to leading order, is

$$\frac{\partial \bar{A}}{\partial \bar{t}_1} - q\frac{\partial^2 \bar{A}}{\partial \bar{Z}^2} = g_0\bar{\tau}_1\bar{A} + J_0\bar{A}\int_0^{+\infty}\eta^{-\frac{1}{2}}[\bar{A}(\bar{t}_1-\eta)\bar{A}_{\bar{Z}}^*(\bar{t}_1-\eta)]_{\bar{Z}}\,\mathrm{d}\eta . \qquad (6.4)$$

For the singular critical layers, we introduce $A = \lambda^{\frac{1}{2}}\bar{A}$ while $\bar{Z}$ is defined as in (6.3). The rescaled form of (6.2), to leading order, becomes

$$\frac{\partial \bar{A}}{\partial \bar{t}_1} - q\frac{\partial^2 \bar{A}}{\partial \bar{Z}^2} = g_0\bar{\tau}_1\bar{A} + \alpha^2 I_0|\bar{A}|^2\bar{A} . \qquad (6.5)$$

Note that it is impossible to rescale (6.2) such that the two nonlinear terms are retained in the equation at the same time. This implies that in the very viscous limit flows with regular critical layers behave quite differently from those with singular critical layers.

We now examine in some detail how the flow structure changes in this viscous limit. Because the critical layers now become steady and viscous-dominated, for the mean-flow distortion the balance within the critical layers is between the viscous diffusion and the Reynolds stress. As such the mean-flow becomes unbounded at the edge of the critical layer and cannot match directly with the flow in the outer layer. In particular, as $Y \to \pm\infty$, the streamwise component grows like $Y^3 \log Y$. Therefore surrounding the critical layer a diffusion layer with an $O(\lambda^{\frac{1}{3}}R^{-\frac{1}{3}})$ width must be introduced to inhibit the unbounded growth (cf. Wu *et al.* 1993; Wu 1993; see also Brown & Stewartson 1978 for the diffusion layer of a purely two-dimensional wave). The balance in the diffusion layer is between the unsteadiness and the viscous diffusion, and the transverse variable is $\tilde{Y} = \lambda^{-\frac{1}{3}}Y$. The mean-flow components have the expansions

$$U_m = \epsilon^{\frac{6}{5}}\lambda\log\lambda^{\frac{1}{3}}\,\tilde{U}_1 + \epsilon^{\frac{6}{5}}\lambda\tilde{U}_2 + \text{c.c.} + \dots ,$$
$$V_m = \epsilon^{\frac{8}{5}}\lambda^{\frac{1}{3}}\log\lambda^{\frac{1}{3}}\,\tilde{V}_1 + \epsilon^{\frac{8}{5}}\lambda^{\frac{1}{3}}\,\tilde{V}_2 + \text{c.c.} + \dots ,$$

$$W_m = \epsilon \log \lambda^{\frac{1}{3}} \tilde{W}_1 + \epsilon \tilde{W}_2 + \text{c.c.} + \dots .$$

The functions $\tilde{U}_2$, $\tilde{V}_2$ and $\tilde{W}_2$ satisfy

$$\frac{\partial \tilde{V}_2}{\partial \tilde{Y}} + \frac{\partial \tilde{W}_2}{\partial Z} = 0 , \tag{6.6}$$

$$\frac{\partial \tilde{U}_2}{\partial \bar{t}_1} - \frac{\partial^2 \tilde{U}_2}{\partial \tilde{Y}^2} + \bar{U}_y \tilde{V}_2 = 0 , \quad \frac{\partial \tilde{W}_2}{\partial \bar{t}_1} - \frac{\partial^2 \tilde{W}_2}{\partial \tilde{Y}^2} = \alpha^{-2} A A_Z^{*} \tilde{Y}^{-2} . \tag{6.7}$$

The matching with the solution in the critical layer requires that as $\tilde{Y} \to 0$,

$$\tilde{U}_2 \to -\tfrac{1}{6}\alpha^{-2}[A^{*}A_Z]_Z \tilde{Y}^3 \log \tilde{Y} , \quad \tilde{V}_2 \to -\alpha^{-2}[A^{*}A_Z]_Z \tilde{Y} \log \tilde{Y} , \quad \tilde{W}_2 \to \alpha^{-2} A^{*} A_Z \log \tilde{Y} . \tag{6.8}$$

The functions $\tilde{U}_1$, $\tilde{V}_1$ and $\tilde{W}_1$ are governed by the same equations as (6.6)–(6.7) provided that the forcing term in the second equation of (6.7) is removed. The matching conditions are

$$\tilde{U}_1 \to -\tfrac{1}{6}\alpha^{-2}[A^{*}A_Z]_Z \tilde{Y}^3 , \quad \tilde{V}_1 \to -\alpha^{-2}[A^{*}A_Z]_Z \tilde{Y} , \quad \tilde{W}_1 \to \alpha^{-2} A^{*} A_Z . \tag{6.9}$$

We find that the interaction between the fundamental and $\tilde{U}_1$, $\tilde{V}_1$, $\tilde{W}_1$ (and their complex conjugates) does not produce any jump, and hence it does not contribute any nonlinear term to the evolution equation. However $\tilde{U}_2^{*}$ interacts with the fundamental to produce the non-local term in (6.2).

The diffusion layer bears some resemblance to the 'buffer layer' of Hall & Smith (1990) and Smith & Walton (1989) in that both regions are introduced to render the (spanwise-dependent) mean flow bounded so that a matching between different zones can be achieved. Also the induced mean flow in the diffusion layer has a much larger magnitude than the harmonics, a feature characteristic of wave–vortex interaction in Smith & Walton (1989), Hall & Smith (1990); see also Hall & Smith (1989), Smith & Blennerhassett (1992), Brown *et al.* (1993) and Smith *et al.* (1993). Indeed, the governing equations that Hall & Smith (1990) derive for their buffer layer are essentially the same as (6.6) and (6.7). However, for T-S waves, the lowest-order mean-flow distortion, i.e. $\tilde{U}_1$ in our notation, can affect the nonlinear interaction (in the lower deck) by producing an alteration to the wall shear. The buffer-layer equations of Smith & Walton (1989) are similar to (6.6) and (6.7) except that $\partial/\partial\bar{t}_1$ is effectively replaced by $\tilde{Y}\partial/\partial x_1$ because the spatial development (over $x_1$) is considered. This difference leads to a kernel $\eta^{-\frac{1}{3}}$ rather than $\eta^{-\frac{1}{2}}$ in their amplitude equation.

### 6.3. *Terminal solutions: focusing singularity*

There are several candidates for the terminal solution of (5.10). For singular critical layers, i.e. $\hat{s}_0 \neq 0$, the first possibility is the two-dimensional equilibrium solution:

$$A = \breve{A}_0 e^{ik\bar{t}} , \tag{6.10}$$

where $\breve{A}_0$ and $k$ are complex and real numbers respectively. This terminal form, if it occurs, implies that the small three-dimensional 'imperfection' finally dies out. However the equilibrium state (6.10) may be subject to a three-dimensional secondary instability formulated by Wu (1991) and hence seems unlikely to be attainable. The second option is that the solution blows up at a finite time, say $t_s$, in the whole flow field, i.e. as $\bar{t} \to t_s$,

$$\bar{A} \to \frac{a_0}{(t_s - \bar{t})^{\frac{5}{2}+i\sigma}} , \tag{6.11}$$

where $\sigma$ and $a_0$ are real and complex numbers respectively and can be determined in the same way as described by Wu & Cowley (1993). The above two forms are associated only with the 'two-dimensional nonlinear term', and can possibly exist only for singular critical layers; three-dimensionality, however, does not play any role in their final formation.

A terminal form of special interest is that the solution blows up at a particular spanwise location, say $Z_s$, within a finite time, say $t_s$. We refer to this as 'focusing singularity'. The structure is proposed as follows:

$$\bar{A} = (t_s - \bar{t})^{-\frac{5}{2}-i\sigma} F(\hat{Z}) \,, \tag{6.12}$$

where

$$\hat{Z} = (t_s - \bar{t})^{-\frac{1}{2}} (\bar{Z} - Z_s) \,. \tag{6.13}$$

A variable of the form (6.13) was introduced in studies such as Hocking *et al.* (1972), and Hall & Smith (1990). Inserting (6.12) into (5.10), we obtain an integro-differential equation for $F(\hat{Z})$, namely

$$\begin{aligned} &-(1+i\bar{q})F''(\hat{Z}) + \tfrac{1}{2}\hat{Z}F'(\hat{Z}) + (\tfrac{5}{2}+i\sigma)F(\hat{Z}) \\ &= \hat{s}_0\bar{g}\int_0^{+\infty}\!\!\int_0^{+\infty} \xi^2 \Theta_0(\xi,\eta) F\left[\frac{\hat{Z}}{(1+\xi)^{\frac{1}{2}}}\right] F\left[\frac{\hat{Z}}{(1+\xi+\eta)^{\frac{1}{2}}}\right] F^*\left[\frac{\hat{Z}}{(1+2\xi+\eta)^{\frac{1}{2}}}\right] d\xi d\eta \\ &+\bar{h}\int_0^{+\infty}\!\!\int_0^{+\infty} \Theta_0(\xi,\eta)\left\{\xi^3 F\left[\frac{\hat{Z}}{(1+\xi)^{\frac{1}{2}}}\right] F\left[\frac{\hat{Z}}{(1+\xi+\eta)^{\frac{1}{2}}}\right] F^*_{\hat{Z}\hat{Z}}\left[\frac{\hat{Z}}{(1+2\xi+\eta)^{\frac{1}{2}}}\right]\right. \\ &+\xi^2\eta F\left[\frac{\hat{Z}}{(1+\xi)^{\frac{1}{2}}}\right]\left\{F\left[\frac{\hat{Z}}{(1+\xi+\eta)^{\frac{1}{2}}}\right] F^*_{\hat{Z}}\left[\frac{\hat{Z}}{(1+2\xi+\eta)^{\frac{1}{2}}}\right]\right\}_{\hat{Z}} \\ &\left.+\xi^3\left\{F\left[\frac{\hat{Z}}{(1+\xi)^{\frac{1}{2}}}\right] F\left[\frac{\hat{Z}}{(1+\xi+\eta)^{\frac{1}{2}}}\right] F^*_{\hat{Z}}\left[\frac{\hat{Z}}{(1+2\xi+\eta)^{\frac{1}{2}}}\right]\right\}_{\hat{Z}}\right\} d\xi d\eta \,, \end{aligned} \tag{6.14}$$

where $\bar{g} = \sum v_j$, $\bar{h} = \sum \bar{h}_j$ and

$$\Theta_0(\xi,\eta) = (1+\xi)^{-\frac{5}{2}-i\sigma}(1+\xi+\eta)^{-\frac{5}{2}-i\sigma}(1+2\xi+\eta)^{-\frac{5}{2}+i\sigma} \,. \tag{6.15}$$

Equation (6.14) must be solved under certain boundary conditions. A simple one is that the disturbance is symmetric about $Z_s = 0$, and decays to zero at infinity, i.e. $F'(0) = 0$ and $F \to 0$ as $\hat{Z} \to +\infty$. Note that the 'two-dimensional nonlinear term' also contributes to the formation of the singularity. But we do not think its contribution is crucial, i.e. even if $\hat{s}_0 = 0$, the singularity may well occur. Our numerical result suggests that this is indeed the case.

While (6.14) is complicated, as $|\hat{Z}| \ll 1$ it has the local solution

$$F = \frac{F_0}{(1+\chi\hat{Z}^2)^{\frac{5}{2}+i\sigma}} \,, \tag{6.16}$$

where $\chi$ and $F_0$ are real and complex constants respectively. This form follows from the consideration that as $t_1 \to t_s$, the amplitude $A$ should remain bounded except at $Z = Z_s = 0$. A solution of similar form also appeared in Hocking *et al.* (1972) and Hocking & Stewartson (1972), The constants $\chi$ and $F_0$ can be determined, in principle, by solving (6.14). Unfortunately, the nonlinear eigenvalue problem posed by (6.14) and the boundary conditions are rather awkward to solve. So we shall take a short cut by choosing $\chi$ such that (6.16) can best fit the numerical solution (see below). Since the amplitude equation does not seem to have been derived before, we wish to bring it to attention by demonstrating its usefulness while leaving the complete resolution of the singularity to the future.

### 6.4. *Numerical solutions of the modulation equation*

In this section, we solve the modulation equation numerically using a finite-difference method. Since there have been detailed experiments for free shear layers, we study the amplitude equation (A 18), so that a qualitative comparison can be made with the observations. We shall concentrate on the case where the initial disturbance is centred at $\bar{Z} = 0$. As noted by Hocking & Stewartson (1972), while the initial disturbance of the form (5.7) is theoretically representative, it is not convenient for numerical integration of the amplitude equation. So following them, we chose the initial condition of the form

$$\bar{A}(\bar{Z},0) = \Xi_0 \exp(-\frac{\bar{Z}^2}{\Lambda_0}) \,. \tag{6.17}$$

This initial distribution is rather arbitrary. To give a realistic distribution, one must resolve a receptivity problem, which is beyond the scope of this paper. Nevertheless, we believe that the mechanism and the qualitative features can be revealed by studying this particular form of disturbance. The parameters $\Xi_0$ and $\Lambda_0$ represent the initial amplitude and the spanwise length-scale respectively. We shall investigate their effects on the formation of the three-dimensional structure.

The finite-difference scheme we use is the standard Crank–Nicolson method. Because of the symmetry, we only need to solve the equation in the domain $\bar{Z} > 0$. The boundary conditions are that $\bar{A}_{\bar{Z}}(0,\bar{x}) = 0$, and $\bar{A} \to 0$ as $\bar{Z} \to \infty$. In calculation, we put $\bar{A} = 0$ when $\bar{Z} \geqslant Z_e$, where we chose $Z_e$ to be a large number so that a change of $Z_e$ does not alter the results appreciably. We find that $Z_e = 20$ is sufficiently large. The spanwise space step $\Delta\bar{Z} = 0.05$ and the 'time' step $\Delta\bar{x} = 0.025$ are used, even though the grid size $\Delta\bar{Z} = 0.1$, $\Delta\bar{x} = 0.05$ is already fine enough.

The first example is for $\kappa = 0.2$. This value is deliberately chosen because it corresponds to the experiment of LCM, though their mean-flow profile may not exactly be the 'tanh' form. The other parameters are $\Xi_0 = 0.1$ and $\Lambda_0 = 4$. For the inviscid case, i.e. $\lambda = 0$, the development of $|\bar{A}|$ is displayed in perspective in figure 2(*a*). It can be seen that the small disturbance is amplified as it is convected downstream. A more notable feature is that, though the initial distribution consists of a single 'hill', more 'hills' are generated downstream. Up to the streamwise location investigated, we can observe at least seven hills (recall the symmetry about $\bar{Z} = 0$). The distances between two neighbouring hills are more or less the same, i.e. the spanwise structure is quasi-periodic (in a finite $\bar{Z}$-range). The appearance of these hills indicates that the streamwise vorticity, which is proportional to $\bar{A}_{\bar{Z}}$, has attained an appreciable strength, and that at the same time, the mean-flow is distorted into a 'peak-valley-splitting' pattern. Further downstream, $|\bar{A}|$ becomes so large and irregular that we can no longer obtain reliable results. This seems to indicate that a singularity is formed at a finite distance downstream. It is postulated that this singularity appears at the spanwise location where $|\bar{A}|$ is a local maximum, and has the structure (6.12) and (6.13). However, we have not been able to verify this for the present situation.

To examine viscous effects, we investigate the case with $\lambda = 5.0$, while all other parameters remain the same as those in figure 2(*a*). As shown in figure 2(*b*), the lateral structure again emerges. However, compared with the inviscid case, the termination of computation is postponed by two units. As a result, up to nine hills are formed.

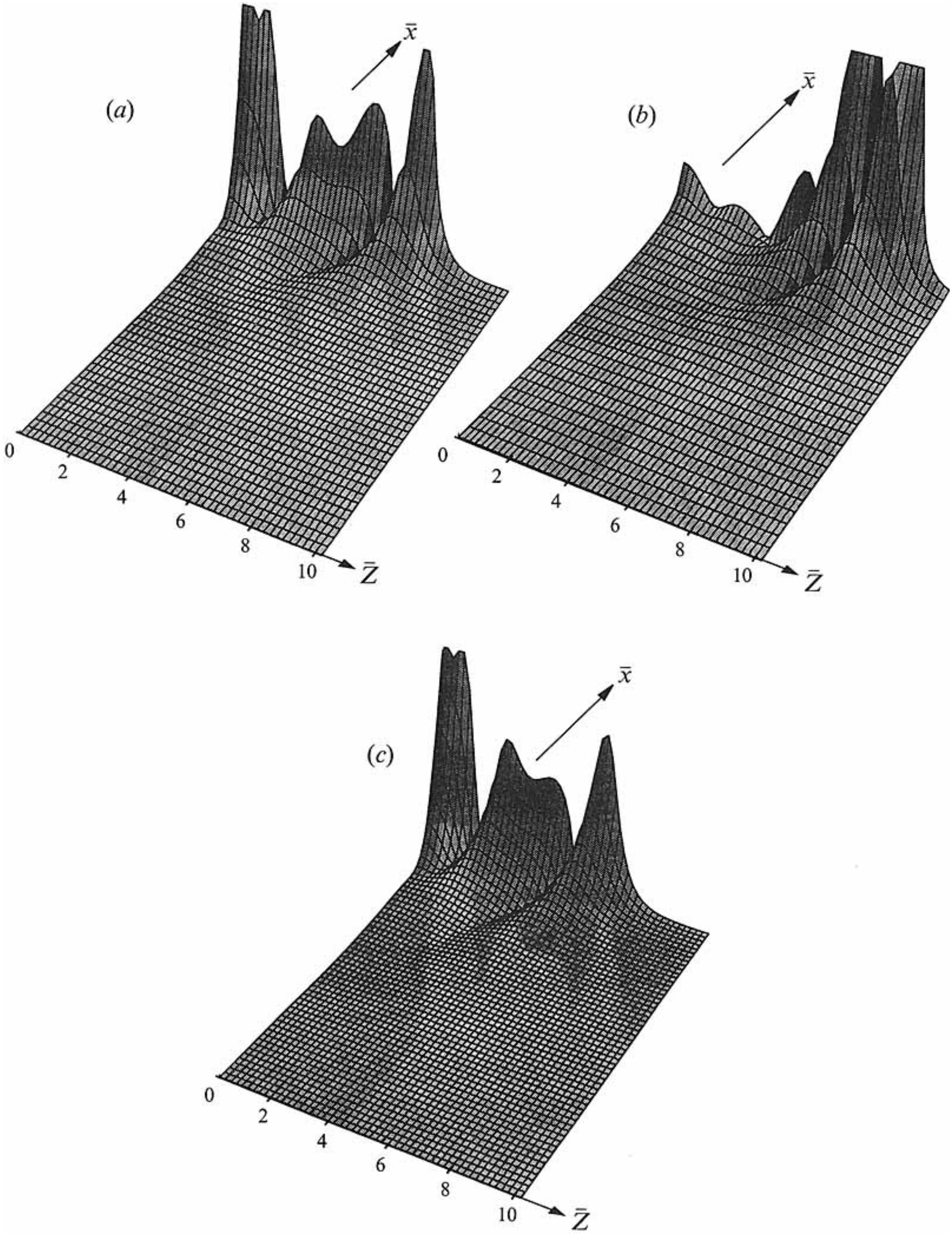


FIGURE 2. (*a–c*). For caption see facing page

As a partial study of the effect of the initial condition on the formation of streamwise vortices, we increase the 'initial strength' $\Xi_0$ to 0.2; the rest of the parameters are the same as in figure 2(*a*). As we can seen from figure 2(*c*), the overall feature remains the same. Nevertheless, compared with figure 2(*a*), the location at which the same number of hills is observed is shifted upstream by approximately two units. This

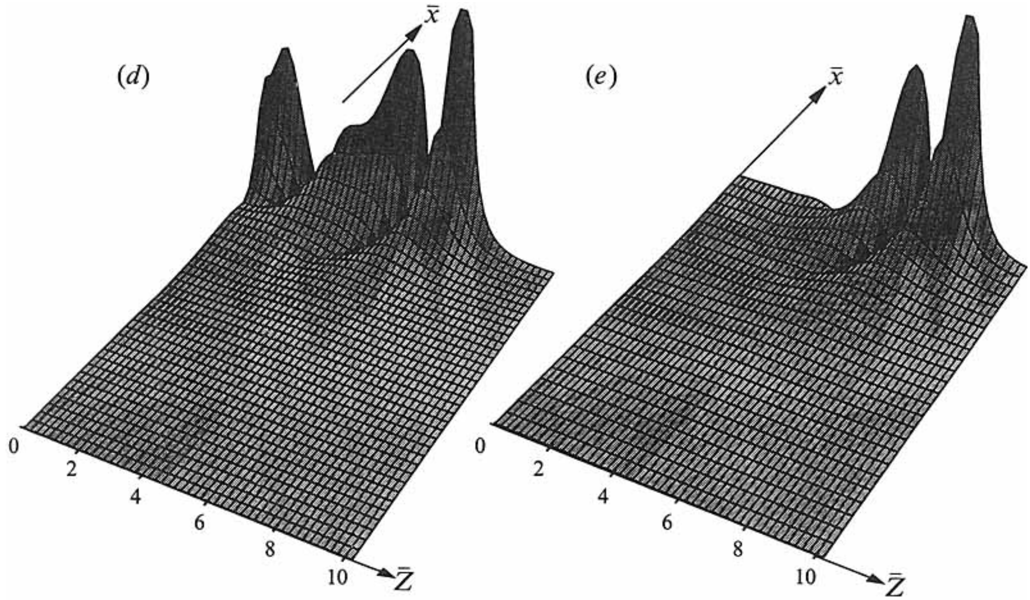


FIGURE 2. A perspective view of the evolution of $|\bar{A}|$: (*a*) $\kappa = 0.2$, $\lambda = 0$ (inviscid limit), $\Xi_0 = 0.1$ and $\Lambda_0 = 4.0$; (*b*) $\kappa = 0.2$, $\lambda = 5.0$ (viscous case), $\Xi_0 = 0.1$ and $\Lambda_0 = 4.0$; (*c*) $\kappa = 0.2$, $\lambda = 0$ (inviscid limit), $\Xi_0 = 0.2$ and $\Lambda_0 = 4.0$; (*d*) $\kappa = 0.1$, $\lambda = 0$ (inviscid limit), $\Xi_0 = 0.1$ and $\Lambda_0 = 4.0$; (*e*) $\kappa = 0.1$, $\lambda = 5.0$, $\Xi_0 = 0.1$ and $\Lambda_0 = 4.0$.

implies that an increase of initial amplitude can cause the streamwise vortices to form earlier. This conclusion is supported by other calculations which are not presented here.

Comparing with the experiment of LCM, we find that our theory can qualitatively capture the observed phenomenon that a disturbance centred at a spanwise location can spread laterally to form a spanwise quasi-periodic structure. The result that this process depends on the initial disturbance is also in consistent with observations (LCM; Lasheras & Choi 1988; Nygaard & Glezer 1991).

We also studied the case $\kappa = 0.1$. The results are depicted in figures 2(*d*) and 2(*e*) for $\lambda = 0$ (the inviscid case) and $\lambda = 5$ (the viscous case) respectively. The development exhibits similar features. So there seems little doubt that the phenomenon predicted is a generic property of our amplitude equation.

To examine more closely the process of the formation of the lateral structure, in figure 3 we plot the spanwise distribution of $|\bar{A}|$ at different streamwise locations. Up to $\bar{x} = 5.6$ (figure 3*a*), the spanwise distribution is still monotonic. Nevertheless, the amplitude is amplified by a factor of 1000. As it evolves downstream, the 'top' of the hill is elongated in the $\bar{Z}$-direction and becomes flattened. At $\bar{x} = 7.2$, the hill splits into two (figure 3*b*). This may correspond to vortex splitting; we note that Bell & Mehta (1992) observed that vortex splitting indeed occurred in the early stage of the evolution. As the hills grow in height, the inner side of each hill gradually develops into a 'valley' (figure 3*c*). Further downstream each valley then becomes deeper, and a new hill is created in between (figure 3*d*). From $\bar{x} = 8.4$, each 'valley' starts to distort, and splits into two (figure 3*e*). The total number of hills at this station increases to 5. The newly created hills continue to grow to a height comparable to that of the

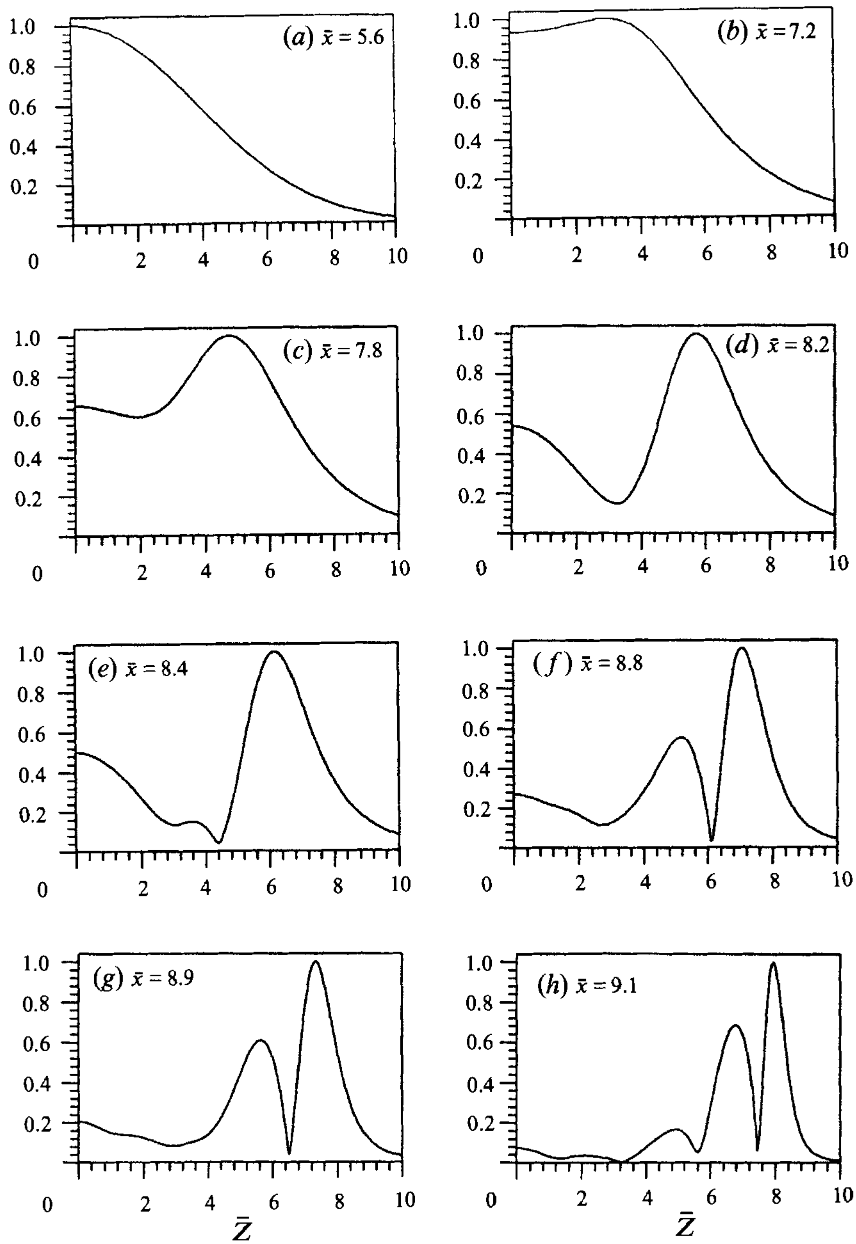


FIGURE 3. Lateral propagation of a localized disturbance: the evolution of spanwise distribution of $|\bar{A}|$ normalized by the maximum at each streamwise location. The parameters are the same as in figure 2(*b*).

outermost one (figure 3*f*); the latter at the same time propagates outward laterally. From the location $\bar{x} = 8.8$, each side of the central hill undergoes a wavy distortion (figure 3*g*), which may be related to the lateral 'undulation' observed by Jimenez (1983) and Nygaard & Glezer (1991). As a result, *two* small hills are generated almost spontaneously on each side (figure 3*h*). The outermost hills further move sideways, exhibiting a 'wave-like' propagation, (though our amplitude equation is of

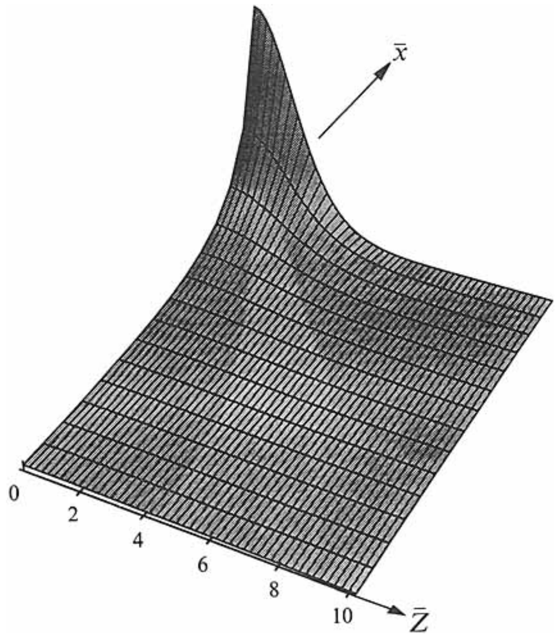


FIGURE 4. A perspective view of the evolution of $|\bar{A}|$ for an artificial case where the coefficient of the nonlinear term in (A18) is changed to $(-1+\kappa)$. The parameters are the same as in figure 1.

diffusion type). It may be instructive to relate this lateral propagation to diffusive waves since it is well known that nonlinear diffusion equations can allow for wave solutions. We terminate our computation at $\bar{x} = 9.1$ although it can be continued slightly downstream. We also examined other cases and find that the process is largely similar.

In all the cases studied above, the coefficient of the nonlinear term is $(1+\mathrm{i}\kappa)$, i.e. the real part is positive. Next we present some results for an artificial case, in which the coefficient is changed to $(-1+\mathrm{i}\kappa)$, while other parameters remain the same as those in figure 2(*a*). As shown in figure 4, the lateral structure does not emerge in this case. Instead, the (single) hill becomes higher and steeper, indicating an energy concentration towards the symmetry plane $\bar{Z} = 0$. This seems to suggest that the sign of the coefficient of the nonlinear term plays an important role in determining whether the lateral structure forms or not. The terminal state appears to be described by (6.12). Using (6.12) or (6.16), we can show that on the symmetry plane $\bar{Z} = 0$ , $|\bar{A}|$ behaves like

$$|\bar{A}(\bar{x},0)| \rightarrow \frac{|F(0)|}{(x_s-\bar{x})^{\frac{5}{2}}} \quad \text{as} \quad x \rightarrow x_s$$

Here $x_s$ is the streamwise location at which the singularity occurs. Using this relation, we estimate from the numerical solution that $x_s \approx 9.2$. In figure 5, we plot the normalized $|\bar{A}|$ against the similarity variable $\hat{Z}$ at different streamwise locations. The dotted line is drawn according to (6.16) with $\chi = 0.2$; this value is chosen so that as $\bar{x} \rightarrow x_s$, the numerical solutions can best fit (6.16). Although this procedure is not completely rigorous, the result displayed in figure 5 does suggest that the proposed singularity may occur in this case. However, we are aware that more study is needed to resolve the proposed singularity in a satisfactory manner.

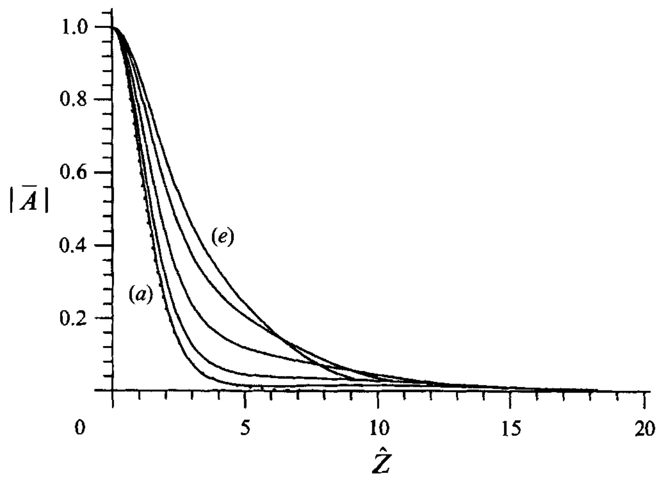


FIGURE 5. A comparison of numerical and the asymptotic solutions: (*a*) $\bar{x} = 7.0$, (*b*) $\bar{x} = 7.3$, (*c*) $\bar{x} = 7.6$, (*d*) $\bar{x} = 7.8$ and (*e*) $\bar{x} = 7.9$. The dotted line is the solution (6.16).

## 7. Conclusions and discussion

### 7.1. *Main results of this study*

In this paper, we have studied the nonlinear evolution of near-planar disturbances in shear flows. A novel modulation equation is obtained which is applicable to broad class of inviscidly unstable shear flows.

In particular, for a free shear layer the numerical solutions show that our amplitude equation can qualitatively capture the formation and the development of the lateral structure from an initially localized disturbance. The qualitative predictions are consistent with the experiment of LCM. Our analysis shows that under the effects of the critical-layer nonlinearity, streamwise vortices can form at a streamwise location upstream of the linear neutral position by a distance of order $\epsilon^{\frac{2}{3}}R$. This is prior to the first spanwise roll-up, which occurs at a position upstream of the neutral point by a distance of $O(\epsilon^{\frac{1}{2}}R)$ (see e.g. Goldstein & Leib 1988). Thus our theory is relevant to the experiments of Lasheras & Choi (1988), Nygaard & Glezer (1991) and LCM in particular, since in these experiments streamwise vortices were observed to form before the first roll-up of the primary spanwise vortices was completed. We note that Bell & Mehta (1992) found that streamwise vortices formed just downstream of the roll-up while Jimenez (1983) found that vortices appeared at the stage where the first pairing of the spanwise vortices occurred. This disagreement may be due to different upstream conditions and different techniques used to detect the structure (see below).

Unlike previous theoretical studies, which relied heavily on numerical computations, our theory is simpler, and much less computationally intensive (though it is still not a trivial task to solve the amplitude equation). The second advantage of our theory is that it tackles physically realistic flows directly. Note that the expressions for the coefficients are valid for any shear flows, though in this paper we have only computed the coefficients explicitly for the 'tanh' shear layer. The theory of Lin & Corcos (1984) and Pullin & Jacobs (1986), however, only models the physics taking place in the braid regions, while Ashurst & Meiburg (1988) used vortex sheets as an ideal model of a shear flow. Finally, in our theory both spatial and temporal development can be considered and are equally convenient. For free shear layers, we studied *spatial*, rather than temporal, evolution, and thus our study is closely relevant to practical situations. It appears that this is the first theoretical study which is able to reveal the development of the lateral structure in the spatial sense. Of course, the validity of our theory is restricted to the case where the initial flow is basically two-dimensional. Once the spanwise distribution is significantly distorted, a new theory must be developed to replace the present one. Alternatively, if the initial spanwise distortion has a short lengthscale, our theory would be largely irrelevant. But we note that an extension of the work by Goldstein & Choi (1989) or Wu *et al.* (1993), which involves a pair of oblique waves, may be able to take account of this situation. Since the subsequent evolution depends on the earlier history, an understanding of the development of the three-dimensional structure from a predominantly two-dimensional flow is of practical importance if this structure is to be efficiently controlled. In this paper we have demonstrated that even a slight three-dimensionality can have a significant effect on the development of the disturbance.

### 7.2. *Further discussion*

To aid the discussion, we now re-examine the nonlinear interaction from the vorticity dynamics point of view. Let $\Omega = (\Omega^{(x)}, \Omega^{(y)}, \Omega^{(z)})$ be the vorticity of the disturbance; then within the critical layer, its components can be expanded as

$$\Omega^{(x)} = \epsilon^{\frac{2}{5}}\Omega_1^{(1)}E + \epsilon^{\frac{3}{5}}\Omega_2^{(x)}(Y,t_1,Z) + \text{c.c.} + \dots ,$$

$$\Omega^{(y)} = \epsilon^{\frac{4}{5}}\Omega_1^{(y)}E + \epsilon^{\frac{7}{5}}\Omega_2^{(y)}(Y,t_1,Z) + \text{c.c.} + \dots ,$$

$$\Omega^{(z)} = \epsilon^{\frac{1}{5}}\Omega_1^{(z)}E + \epsilon^{\frac{4}{5}}\Omega_2^{(z)}(Y,t_1,Z) + \text{c.c.} + \dots ,$$

where $\Omega_1^{(x)} = \partial \hat{W}_1/\partial Y$, $\Omega_1^{(y)} = -\mathrm{i}\alpha \hat{W}_1$ and $\Omega_1^{(z)} = -\partial \hat{U}_1/\partial Y$ to the order of approximation. The vorticity components associated with the induced mean-flow, $\Omega_2^{(x)}$, $\Omega_2^{(y)}$ and $\Omega_2^{(z)}$, satisfy

$$\frac{\partial \Omega_2^{(x)}}{\partial t_1} - \lambda \frac{\partial^2 \Omega_2^{(x)}}{\partial Y^2} = -\hat{A}\frac{\partial \Omega_1^{*(x)}}{\partial Y} , \tag{7.1}$$

$$\frac{\partial \Omega_2^{(y)}}{\partial t_1} - \lambda \frac{\partial^2 \Omega_2^{(y)}}{\partial Y^2} = -\hat{A}\frac{\partial \Omega_1^{*(y)}}{\partial Y} + (-\Omega_1^{*(z)})\frac{\partial \hat{A}}{\partial Z} + \bar{\Omega}_b \frac{\partial \hat{V}_2^{(0)}}{\partial Z} , \tag{7.2}$$

$$\frac{\partial \Omega_2^{(z)}}{\partial t_1} - \lambda \frac{\partial^2 \Omega_2^{(z)}}{\partial Y^2} = -\hat{A}\frac{\partial \Omega_1^{*(z)}}{\partial Y} + \bar{\Omega}_b \frac{\partial \hat{W}_2^{(0)}}{\partial Z} , \tag{7.3}$$

where $\bar{\Omega}_b = -\bar{U}_y(y_c)$ is the shear or the spanwise vorticity of the basic flow. The first term on the right-hand side of (7.2) comes from $(\boldsymbol{u}\cdot\nabla)\Omega$, where $\boldsymbol{u}$ is the instantaneous velocity field. The second term on the right-hand of (7.2) arises from $(\Omega\cdot\nabla)\boldsymbol{u}$, and represents the stretching of the primary spanwise vorticity by the strain $\partial A/\partial Z$ which is created because of non-uniformality in the spanwise direction. Such a stretching contributes to produce a vorticity component in the normal direction, i.e. $\Omega_2^{(y)} \equiv \partial \hat{U}_2^{(0)}/\partial Z$. The distribution of this quantity in the $z$-direction faithfully reflects the change of the spanwise lengthscale and the development of the three-dimensional structure, as suggested by Nygaard & Glezer (1991). The Reynolds stress in (7.1) is from $(\boldsymbol{u}\cdot\nabla)\Omega$, rather than from $(\Omega\cdot\nabla)\boldsymbol{u}$. This indicates that the streamwise vorticity is not directly driven by stretching, nor itself is stretched, contrary to the Lin–Corcos model. The reason is that the disturbance that we are considering has a large spanwise lengthscale. Nevertheless, the stretching of the primary spanwise vorticity affects the amplitude and hence contributes to the generation of $\Omega_2^{(x)}$ through the forcing term in (7.1). The last terms in (7.2) and (7.3) can be interpreted as the stretching of the vorticity of the basic flow by the strain $\partial \hat{V}_2^{(0)}/\partial Z$ and $\partial \hat{W}_2^{(0)}/\partial Z$, which arise again as a result of non-uniformality in the $z$-direction. Since in the Lin–Corcos (1984) model, the external strain field is completely depleted of vorticity, this effect is absent there. Therefore, while our analysis indicates somewhat the importance of the stretching associated with three-dimensionality, the way that the stretching is involved differs from those proposed, e.g. by Lin & Corcos (1984), LCM, Lasheras & Choi (1988), Bell & Mehta (1992). Our analysis shows that when the spanwise lengthscale is large, the stretching of primary spanwise vorticity dominates. Moreover, the strain involved is created by spanwise non-uniformality rather than by the spanwise vorticity as in the Lin–Corcos model. We suggest that such stretching may be responsible for the bending of primary (spanwise) vortex rolls and the spanwise undulation observed, e.g. by Nygaard & Glezer (1991). While our study implies that longitudinal vortices can be 'initiated' without stretching the streamwise vorticity, we must point out that once the spanwise lengthscale becomes sufficiently short (as a result of the distortion), the streamwise vorticity stretching will become important and possibly dominant.

Based on the results of this paper and the discussion above, a possible mechanism for the generation of streamwise vortices can be stated as follows. A basically two-dimensional disturbance is slightly distorted in a three-dimensional manner by some unavoidable small imperfection present in the flow. Owing to this non-uniformity in the spanwise direction, a strain field is created. As the two-dimensional wave is amplified, the strain grows in strength. The primary spanwise vorticity is then continuously stretched. This may be the main nonlinear activity leading to the redistribution of energy and vorticity in the spanwise direction and the induction of other vorticity components including streamwise vorticity. The spanwise redistribution is manifested in the formation of quasi-periodic streamwise vortices. It is clear that the growing of the primary spanwise vortices is crucial in our approach because it is this growth that brings nonlinearity into play.

It appears to be necessary to clarify the concept of 'secondary instability', which has been frequently invoked to explain the formation of the streamwise vortices. On the one hand, their origin is certainly associated with some form of instability mechanism. More precisely, it is 'the result of the unstable response of the layer to the three-dimensional perturbation in the upstream condition', as concluded by LCM. On the other hand, owing to the growth of the two-dimensional flow, it seems inappropriate to formulate this secondary instability as an eigenvalue problem, and then to isolate any single mode responsible for the observed three-dimensional structure. Instead, it should be formulated into an initial-value problem† (see also Corcos & Lin 1984). The support for this view point can be found from previous studies. For example, Ashurst & Meiburg (1988) concluded that their numerical simulations 'did not reveal any single disturbance which causes the three-dimensional instability of the plane mixing layer'. Nygaard & Glezer (1991) found that 'any spanwise wavelengths synthesizable by the heating mosaic [used to generate a spanwise disturbance] can be excited, and can lead to the formation of streamwise vortices'. Lasheras & Choi (1988) and LCM found that there was no most unstable mode. We note that Pierrehumbert & Widnall (1982) identified translative modes for Stuart vortices. However, this became possible because a steady, rather than a growing, basic flow was used. It appears unlikely that the development of the streamwise vorticity is associated with the excitation

† This is not dissimilar to the situation of Görtler instability; see e.g. Hall (1991) and references therein.

of any particular mode (Lasheras & Choi 1988; Ashurst & Meiburg 1988). Given that the process is described by an initial-value instability problem, the subsequent development then tends to depend on the 'initial' conditions. Therefore it is not surprising that discrepancies exist regarding the spanwise spacing, the streamwise location where the longitudinal streaks are first formed, etc. since they are affected by upstream conditions. This have been strongly suggested by many experiments, e.g. Jimenez (1983), Jimenez *et al.* (1985), LCM, Lasheras & Choi (1988), Nygaard & Glezer (1991).

In this paper, we have only solved (A 18), a special case of (5.10). The range of the parameters that we have examined is also limited. Obviously, further numerical study is needed to explore the properties of (5.10). In particular, the solutions to (A 11) and (A 18) with a spanwise-periodic 'initial condition' deserve further investigation. We expect that such a study would provide insight into the phenomenon observed in the experiments of Lasheras & Choi (1988) and Nygaard & Glezer (1991).

Finally, we note that the present theory can be extended to compressible and stratified flows. In addition, by adopting an upper-branch scaling (Bodonyi & Smith 1981), the present approach can be modified to describe the evolution of a T-S wave packet in boundary layers and channel flows (the analysis for which is related to the viscous limit discussed in §6). Such a study would be relevant to the experiment of Gaster & Grant (1975), and may explain the observed distortion, in particular the spanwise splitting of wavepackets. This is currently under investigation by the author in collaboration with Dr S.J. Cowley.

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## Appendix A

In this appendix, by modifying the analysis in the main text of this paper, we deduce the amplitude equation for disturbances in free shear layers.

As stressed in §2, the scaling for free shear layers is the same as that for the Stokes layer, i.e. nonlinear effects first become important when the linear growth rate decreases to $O(\epsilon^{\frac{2}{5}})$. This happens at the position where the local Strouhal number is

$$S = S_0 + \epsilon^{\frac{2}{5}} S_1 , \tag{A 1}$$

where $S_1 < 0$, and $S_0$ is the local Strouhal number at the linear neutral point. Throughout this appendix, the notation for the mean-flow quantities, e.g. $U_c$, $\delta_0$, etc. is the same as in Goldstein & Choi (1989) or Hultgren (1992). The reader is recommended to consult these for definitions. The local Reynolds number $R = \delta_0 \Delta / \nu$ is scaled as (2.10). For free shear layers, it is appropriate to consider the spatial development, and the variable describing streamwise evolution is

$$x_1 = \epsilon^{\frac{2}{5}} U_c^{-1} x . \tag{A 2}$$

Here we have introduced a factor $U_c^{-1}$ in (A 2). This will allow us to make use of the results in the main text of this paper without going through a detailed expansion procedure. The slowly varying spanwise variable is defined (2.14).

Outside the critical layer, the vertical velocity of the disturbance is expanded as

$$v = \epsilon A(Z,x_1)\bar{v}_1(y)E + \epsilon^{\frac{7}{5}}\bar{v}_2(y,x_1,Z)E + \text{c.c.} + \dots , \tag{A 3}$$

where $E = \mathrm{e}^{\mathrm{i}\alpha\zeta}$, and $\zeta = x - U_c t - \epsilon^{\frac{2}{5}} S_1/\alpha t$. The expansions for the pressure and other velocity components are similar to (3.4), (3.1) and (3.3) respectively. The function $\bar{v}_1$ satisfies the Rayleigh equation

$$R_a\bar{v}_1 = 0 , \quad \text{where} \quad R_a = \frac{\mathrm{d}^2}{\mathrm{d}y^2} - (\alpha^2 + \frac{U''}{U-c}) . \tag{A 4}$$

The solution near the critical level $y_c = 0$ is (see e.g. Goldstein & Choi 1989)

$$\bar{v}_1 = 1 + \frac{1}{2}\left(\alpha^2 + \frac{U_c'''}{U_c'}\right) y^2 + b_1 y + O(y^3) . \tag{A 5}$$

The function $\bar{v}_2(y,x_1,Z)$ satisfies the inhomogeneous Rayleigh equation:

$$R_a\bar{v}_2 = -2\alpha\mathrm{i}U_c^{-1}\frac{\partial A}{\partial x_1}\bar{v}_1 - \mathrm{i}\alpha^{-1}\left[\mathrm{i}S_1 A - \frac{\partial A}{\partial x_1}\right]\frac{U''}{(U-c)^2}\bar{v}_1 - \frac{\partial^2 A}{\partial Z^2}\bar{v}_1 . \tag{A 6}$$

As $y \to \pm 0$, (A 6) has the solution

$$\bar{v}_2 = d + c^{\pm}y + \frac{\mathrm{i}U_c'''}{\alpha U_c'^2}\left[\frac{\partial A}{\partial x_1} - \mathrm{i}S_1 A\right] y\log|y| + O(y^2) . \tag{A 7}$$

The solvability condition for (A 6) is

$$c^+ - c^- = 2\mathrm{i}\alpha U_c^{-1}\hat{J}_1\frac{\partial A}{\partial x_1} - \mathrm{i}\alpha^{-1}\left[\frac{\partial A}{\partial x_1} - \mathrm{i}S_1 A\right]\hat{J}_2 + \hat{J}_1\frac{\partial^2 A}{\partial^2 Z} , \tag{A 8}$$

where $\hat{J}_1$ and $\hat{J}_2$ are constants defined by

$$\hat{J}_1 = \int_{-\infty}^{+\infty}\bar{v}_1^2\mathrm{d}y , \qquad \hat{J}_2 = \int_{-\infty}^{+\infty}\frac{U''\bar{v}_1^2}{(U-c)^2}\mathrm{d}y . \tag{A 9}$$

Note that $\hat{J}_2$ should be interpreted as a Cauchy principal value.

Because the critical layer is regular, the solution contains only one jump $(c^+ - c^-)$. It must be determined by analysing the dynamics within the critical layer. So we introduce an inner variable: $Y = y/\epsilon^{\frac{2}{5}}$. Within the critical layer, the expansions take the form of (4.1)–(4.4). The 'critical-layer operator' is

$$L_0 = \left[\frac{\partial}{\partial x_1} + (U_c'Y - S_1/\alpha)\frac{\partial}{\partial x}\right] - \lambda\frac{\partial^2}{\partial Y^2} .$$

Because we have defined $x_1$ by (A 2), this operator is slightly different from that in Goldstein & Choi (1989) and Hultgren (1992), but is exactly the same as (4.5) if we identify $x_1$ as $t_1$, and $S_1/\alpha$ as $\bar{U}_\tau\tau_1$. Therefore we can borrow the result (4.28) to obtain the jump $(c^+ - c^-)$, namely

$$\begin{aligned} c^+ - c^- = &- \frac{\pi U_c'''}{\alpha U_c'^2}\left[\frac{\partial A}{\partial x_1} - \mathrm{i}S_1 A\right] \\ &- 2\pi\alpha|U_c'|^3\int_0^{+\infty}\int_0^{+\infty}K(\xi,\eta)\,\{\xi^3 A(Z,x_1-\xi)A(Z,x_1-\xi-\eta)A^*_{ZZ}(Z,x_1-2\xi-\eta) \\ &+ \xi^2\eta A(Z,x_1-\xi)[A(Z,x_1-\xi-\eta)A^*_Z(Z,x_1-2\xi-\eta)]_Z \\ &+ \xi^3\,[A(Z,x_1-\xi)A(Z,x_1-\xi-\eta)A^*_Z(Z,x_1-2\xi-\eta)]_Z\}\,\mathrm{d}\xi\mathrm{d}\eta . \end{aligned} \tag{A 10}$$

The linear part follows from the $\pi$-phase shift at the logarithmic branch point (see (A 7)). Since for free shear layers, $U_c'' = 0$, the first nonlinear term in (4.28) now disappears. Using (A 8) and (A 10), we obtain the amplitude equation for free shear layers:

$$\frac{\partial A}{\partial x_1} - q\frac{\partial^2 A}{\partial Z^2} - g_0 A$$
$$= h\int_0^{+\infty}\int_0^{+\infty} K(\xi,\eta|\lambda)\,\{\xi^3 A(Z,x_1-\xi)A(Z,x_1-\xi-\eta)A^*_{ZZ}(Z,x_1-2\xi-\eta)$$
$$+\xi^2\eta A(Z,x_1-\xi)[A(Z,x_1-\xi-\eta)A^*_Z(Z,x_1-2\xi-\eta)]_Z$$
$$+\xi^3[A(Z,x_1-\xi)A(Z,x_1-\xi-\eta)A^*_Z(Z,x_1-2\xi-\eta)]_Z\}\,\mathrm{d}\xi\mathrm{d}\eta\,, \quad \text{(A 11)}$$

where

$$K(\xi,\eta|\lambda) = \mathrm{e}^{-s(3\xi^3+2\xi^2\eta)}, \qquad \text{and} \qquad s = \tfrac{1}{3}\lambda\alpha^2 U_c^2\,. \quad \text{(A 12)}$$

The sum, along with the suffix, are omitted since there exists only one critical layer. The coefficients are found to be as follows:

$$q = \alpha\hat{J}_1/f\,, \quad g_0 = -\left[\hat{J}_2 + \frac{\pi \mathrm{i} U_c'''}{U_c'^2}\right] S_1/f\,, \quad \text{(A 13)}$$

$$h = 2\pi\alpha^2|U_c|^3/f\,, \quad f = -\frac{2\mathrm{i}\alpha^2}{U_c}\hat{J}_1 + \mathrm{i}\left[\hat{J}_2 + \frac{\pi\mathrm{i}U_c'''}{U_c'^2}\right]\,. \quad \text{(A 14)}$$

Note that the expressions for the coefficients are given explicitly in terms of the mean-flow profile and the eigenfunction (cf. Goldstein & Choi 1989). They are valid for free shear layers with various profiles. Specifically, for the 'tanh' shear layer, we have (see e.g. Huerre 1987)

$$U_c = \frac{U^{(1)}+U^{(2)}}{2\Delta}\,, \quad U_c' = \alpha = 1\,, \quad U_c''' = -2, \quad \hat{J}_1 = 2, \quad \hat{J}_2 = 0\,. \quad \text{(A 15)}$$

It is this special case that we shall investigate numerically in this paper. We can scale out various parameters by introducing the variables similar to (5.8)–(5.9):

$$\bar{A} = A\mathrm{e}^{-\mathrm{i}(\bar{X}_0+g_{0i}x_1)}|h_r|^{\frac{1}{2}}|q_r|^{-\frac{1}{2}}/(g_{0r})^{\frac{5}{2}}\,, \quad \text{(A 16)}$$

$$\bar{x} = g_{0r}x_1\,, \quad \bar{Z} = (g_{0r})^{\frac{1}{2}}|q_r|^{-\frac{1}{2}}Z\,, \quad \bar{\lambda} = \lambda/(g_{0r})^3\,. \quad \text{(A 17)}$$

The amplitude equation then becomes

$$\frac{\partial\bar{A}}{\partial\bar{x}} - (1+\mathrm{i}\kappa)\frac{\partial^2\bar{A}}{\partial\bar{Z}^2} - \bar{A}$$
$$= (1+\mathrm{i}\kappa)\int_0^{+\infty}\int_0^{+\infty} K(\xi,\eta|\lambda)\,\{\xi^3\bar{A}(\bar{Z},\bar{x}-\xi)\bar{A}(\bar{Z},\bar{x}-\xi-\eta)\bar{A}^*_{\bar{Z}\bar{Z}}(\bar{Z},\bar{x}-2\xi-\eta)$$
$$+\xi^2\eta\bar{A}(\bar{Z},\bar{x}-\xi)[\bar{A}(\bar{Z},\bar{x}-\xi-\eta)\bar{A}^*_{\bar{Z}}(\bar{Z},\bar{x}-2\xi-\eta)]_{\bar{Z}}$$
$$+\xi^3[\bar{A}(\bar{Z},\bar{x}-\xi)\bar{A}(\bar{Z},\bar{x}-\xi-\eta)\bar{A}^*_{\bar{Z}}(\bar{Z},\bar{x}-2\xi-\eta)]_{\bar{Z}}\}\,\mathrm{d}\xi\mathrm{d}\eta\,, \quad \text{(A 18)}$$

with the 'initial condition' $\bar{A} \to \bar{A}(\bar{Z},0)$ as $\bar{x} \to 0$. Here $\bar{\lambda}$ in the kernel is written as $\lambda$, and

$$\kappa = \frac{2}{\pi U_c}\,.$$

## Appendix B

In this appendix, we show that as $\lambda \to +\infty$, the fully integral-partial-differential equation (5.1) reduces to (6.2). For convenience, let $N^{(j)}$ ($j = 1,2,3,4$) denote the first, second, third and fourth nonlinear terms in (5.1) respectively.

We first observe that after substituting the change of variables

$$t_1 = \lambda^{\frac{1}{3}}\bar{t}_1\,, \qquad \xi = \lambda^{-\frac{1}{3}}\bar{\xi}\,, \qquad \eta = \lambda^{-\frac{1}{3}}\bar{\eta}\,, \tag{B 1}$$

into $N^{(1)}$, $N^{(2)}$ and $N^{(4)}$, and taking the limit $\lambda \to +\infty$, we immediately have

$$N^{(1)} \to \lambda^{-\frac{4}{3}}\frac{1}{3^{\frac{2}{3}} \times 2^{\frac{1}{3}}}\Gamma(\tfrac{1}{3})A|A|^2\,, \tag{B 2}$$

$$N^{(2)} \to \lambda^{-\frac{5}{3}}C_0A^2A^*_{ZZ}\,, \tag{B 3}$$

$$N^{(4)} \to \lambda^{-\frac{5}{3}}C_0[A|A_Z|^2 + A^2A^*_{ZZ}]\,, \tag{B 4}$$

where $C_0$ is a constant defined by a convergent integral. Note that these nonlinear terms revert to classical cubic form with history effects being damped out.

To estimate $N^{(3)}$, we integrate $N^{(3)}$ by parts with respect to $\eta$ to obtain

$$\begin{aligned} N^{(3)} = {}&(-3s)^{-1}\int_0^{+\infty}\int_0^{+\infty}\eta K_j(\xi,\eta|\lambda)A(t_1-\xi)\frac{\partial}{\partial t_1}[A(t_1-\xi-\eta)A^*_Z(t_1-2\xi-\eta)]_Z\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &-(3s)^{-2}\int_0^{+\infty}\int_0^{+\infty}\xi^{-2}[\mathrm{e}^{-3s\xi^2\eta}-1]\mathrm{e}^{-3s\xi^3}A(t_1-\xi)\frac{\partial}{\partial t_1}[A(t_1-\xi-\eta)A^*_Z(t_1-2\xi-\eta)]_Z\,\mathrm{d}\xi\,\mathrm{d}\eta\,. \end{aligned} \tag{B 5}$$

We then substitute the change of variables

$$t_1 = \lambda^{\frac{1}{3}}\bar{t}_1\,, \qquad \xi = \lambda^{-\frac{2}{3}}\bar{\xi}\,, \qquad \eta = \lambda^{\frac{1}{3}}\bar{\eta}\,, \tag{B 6}$$

into (B 5) and take the limit $\lambda \to +\infty$; we find

$$N^{(3)} \to \frac{\sqrt{3}}{18}\Gamma(\tfrac{1}{2})\lambda^{-\frac{4}{3}}(\beta_j)^{-\frac{3}{2}}A\frac{\partial}{\partial \bar{t}_1}\left\{\int_0^{+\infty}\bar{\eta}^{\frac{1}{2}}[A(\bar{t}_1-\bar{\eta})A^*_Z(\bar{t}_1-\bar{\eta})]_Z\,\mathrm{d}\bar{\eta}\right\}\,, \tag{B 7}$$

or equivalently

$$N^{(3)} \to \frac{\sqrt{3}}{36}\Gamma(\tfrac{1}{2})\lambda^{-\frac{4}{3}}(\beta_j)^{-\frac{3}{2}}A\int_0^{+\infty}\eta^{-\frac{1}{2}}[A(\bar{t}_1-\eta)A^*_Z(\bar{t}_1-\eta)]_Z\,\mathrm{d}\eta\,, \tag{B 8}$$

where $\beta_j = \frac{1}{3}\alpha^2\bar{U}_y^2(y_c^j)$. Using (B 8), and (B 2) to (B 4), we obtain (6.2).

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